THE LOGIC OF PREDICTION

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Abstract

When we make a non-trivial prediction about the future we select, among the conceivable future descriptions of the world, those that appear to us to be most likely. Within a branching-time framework we capture this by means of two binary relations, $\prec_c$ and $\prec_p$. If t_1 and t_2 are different points in time, we interpret $t_1 \prec_c t_2$ as saying that t_2 is in the *conceivable future* of t_1, while $t_1 \prec_p t_2$ is interpreted to mean that t_2 is in the *predicted future* of t_1. We propose the following notion of "consistency of predictions". Suppose that at t_1 some future moment t_2 is predicted to occur, then (a) every moment t between t_1 and t_2 should also be predicted at t_1 and (b) the prediction of t_2 should continue to hold at every t between t_1 and t_2. We provide a sound and complete axiomatization for this notion of consistency.

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1. Introduction

When we make a non-trivial prediction about the future we select, among the conceivable future descriptions of the world, those that appear to us to be most likely. Thus the concept of prediction involves (at least) three notions: (1) time (predictions are about the future)[1], (2) conceivable future states of the world and (3) a selection from the set of conceivable states of those that are considered likely. We propose a system of modal logic that incorporates these three elements. First of all, the notion of a multiplicity of possible future states is captured by what is known in temporal logic as *branching time* (see, for example, van Benthem, 1991, Burgess, 1984, Goldblatt, 1992, Halpin, 1988, Øhrstrøm and Hasle, 1995). Secondly, to capture the distinction between conceivable and likely future possibilities we introduce two binary relations, $\prec_c$ and $\prec_p$. If t_1 and t_2 are different points in time, we interpret $t_1 \prec_c t_2$ as saying that t_2 is in the *conceivable future* of t_1, while $t_1 \prec_p t_2$ is interpreted to mean that t_2 is in the *predicted future* of t_1.[2]

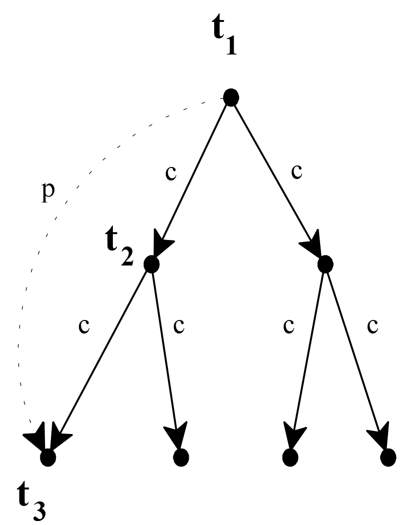

Figure 1

[1] A Chinese proverb says, "Prediction is difficult, especially with regard to the future".

[2] This dual framework is reminiscent of the joint treatment (in another branch of modal logic) of knowledge and belief (see, for example, Halpern, 1991, Hintikka, 1962, van der Hoek, 1993, Kraus and Lehman, 1988, Lenzen, 1978). Indeed making a prediction is essentially expressing a belief about the future. The set of conceivable future states can be thought of as what we "know" about the future, while the set of likely future developments represents what we "believe" about the future.

The main question we address is what "consistency" properties ought to be imposed on the notion of prediction. We propose the following. Suppose that at time t_1 a conceivable future development is represented by the path $t_1t_2t_3$ (that is, $t_1 \prec_c t_2$ and $t_2 \prec_c t_3$). This is shown in Figure 1, where there is a (continuous) arrow labelled 'c' from t to t' if and only if $t \prec_c t'$. Suppose also that t_3 lies in the predicted future of t_1 (that is, $t_1 \prec_p t_3$: this is shown in Figure 1 by a dotted arrow labelled 'p' from $t_1$to t_3). Then we impose the following requirements:

- (a) since reaching t_3 requires going through t_2, t_2 should lie in the predicted future of t_1 (that is, $t_1 \prec_p t_2$), and
- (b) since reaching t_2 is consistent with (is a partial realization of) the prediction that t_3 will be reached, the prediction should continue to hold at t_2, that is, t_3 should be in the predicted future of t_2 ($t_2 \prec_p t_3$).

We provide a sound and complete axiomatization of this notion of time consistency. The semantics is studied in Section 2, while the syntax is developed in Section 3, where the soundness and completeness theorems are proved. Section 4 discusses possible extensions, while Section 5 contains concluding remark.

2. Semantics

We consider what is known in philosophy as *branching time*. Let T be a set of time instants (each instant will be interpreted as a state of the world at a time) and $\prec_c$ and $\prec_p$ binary relations on T. If $t_1 \prec_c t_2$ we say that t_2 is in the *conceivable future* of t_1, while if $t_1 \prec_p t_2$ we say that t_2 is in the *predicted future* of t_1.

Definition 2.1. *A frame is a triple $\mathcal{F} = \langle T, \prec_c, \prec_p \rangle$ where T is a (possibly infinite) set and $\prec_c$ and $\prec_p$ are binary relations on T. A tree-frame is a frame that satisfies the following properties: $\forall t_1, t_2, t_3 \in T$,*

(R.0) antisymmetry of $\prec_c$: if $t_1 \prec_c t_2$ then $t_2 \not\prec_c t_1$,

(R.1) transitivity of $\prec_c$: if $t_1 \prec_c t_2$ and $t_2 \prec_c t_3$ then $t_1 \prec_c t_3$,

(R.2) backward linearity of $\prec_c$: if $t_1 \prec_c t_3$ and $t_2 \prec_c t_3$ then either $t_1 = t_2$ or $t_1 \prec_c t_2$ or $t_2 \prec_c t_1$,

(R.3) $\prec_p$ subrelation of $\prec_c$: if $t_1 \prec_p t_2$ then $t_1 \prec_c t_2$,

(R.4) transitivity of $\prec_p$: if $t_1 \prec_p t_2$ and $t_2 \prec_p t_3$ then $t_1 \prec_p t_3$.

(R.0)-(R.2) constitute the definition of *branching time* in temporal logic.[3] In particular, (R.2) expresses the notion that, while a given moment may have different possible futures, its past is settled. (R.3) captures the notion that predicting the future consists in selecting a subset of the conceivable future states: those that are believed to be most likely. The interpretation of $\prec_p$ in terms of prediction (i.e. belief about the future) makes (R.4) a natural requirement: it can be viewed as incorporating a principle of coherence of belief very close in spirit to van Fraassen's Reflection Principle (van Frassen, 1984). If, now, I consider it possible that at some future time Iraq shoots down a US reconnaissance plane and in a state of affairs where this did happen I would consider it possible that the US would retaliate with an air strike, then I must *now* consider possible a future state of affairs where the US launches a retaliatory air strike on Iraq.[4]

Every $t \in T$ should be thought of as a complete description of the world at time t. Furthermore, sets of dates ought to be thought of as propositions, which in turn are the object of predictions about the future. In order to establish this interpretation one needs to introduce a formal language and the notion of a model based on a frame. This will be done in Section 3.

Given the interpretation of $\{t' \in T : t \prec_p t'\}$ as the "predicted future of t", what further restrictions on the relations $\prec_c$ and $\prec_p$ should one impose? As argued above, the following seems a natural "consistency" requirement: if t_3 is in the predicted future of t_1, and t_2 is on the $\prec_c$-path from t_1 to t_3 then (i) t_2 should be in the predicted future of t_1 and (ii) t_3 should be in the predicted future of t_2. Formally ('CP' stands for 'Consistency of Prediction'),

$$(CP) \quad \forall t_1, t_2, t_3 \in T, \text{ if } \ t_1 \prec_p t_3 \text{ and } t_1 \prec_c t_2 \text{ and } t_2 \prec_c t_3 \\ \text{then } t_1 \prec_p t_2 \text{ and } t_2 \prec_p t_3$$

The following, more general, version of this principle is easily seen to be equivalent to (CP) whenever $\prec_c$ is transitive.[5]

[3] See, for example, Burgess (1984), Halpin (1988), Øhrstrøm and Hasle (1995).

[4] The only result where transitivity of $\prec_p$ is used is Lemma (2.4). Thus it could easily be dropped and the soundness and completeness proofs for the resulting weaker logic would be simplified versions of the proofs given below.

[5] (CP) is a special case of (CP$'$) (the case where $n = 3$). To show that (CP) implies (CP$'$), let $t_1, ..., t_n \in T$ be such that $t_1 \prec_p t_n$ and $t_i \prec_c t_{i+1}$ $\forall\ i = 1, ..., n-1$. Suppose that $n > 3$. By transitivity of $\prec_c$, $t_1 \prec_c t_{n-1}$. Thus by (CP) $t_{n-1} \prec_p t_n$ and $t_1 \prec_p t_{n-1}$. Thus we have reduced to the case $n - 1$. If $n - 1 = 3$, the proof is completed by a second application of (CP). If $n - 1 > 3$ then the argument can be repeated until the sequence is reduced to three elements.

$$(CP') \quad If \;\; t_1, ..., t_n \in T, \;\; are\ such\ that \;\; t_1 \prec_p t_n \;\; and \;\; t_i \prec_c t_{i+1} \;\; \forall\, i = 1, ..., n-1, then \;\; t_i \prec_p t_{i+1} \;\; \forall\, i = 1, ..., n-1$$

Lemma 2.2. *If $\prec_c$ is antisymmetric and backward linear and $\prec_p$ is a subrelation of $\prec_c$ then (CP) is equivalent to the conjunction of the following two properties:*

(R.5) backward linearity of $\prec_p$: if $t_1 \prec_p t_3$ and $t_2 \prec_p t_3$ then either $t_1 = t_2$ or $t_1 \prec_p t_2$ or $t_2 \prec_p t_1$,

(R.6) if $t_1 \prec_p t_3$ and $t_2 \prec_c t_3$ then either (a) $t_1 = t_2$ or (b) $t_2 \prec_c t_1$ or (c) $t_1 \prec_c t_2$ and $t_2 \prec_p t_3$.

Proof. First we prove that (CP) implies (R.5). Let $t_1 \prec_p t_3$ and $t_2 \prec_p t_3$. Since $\prec_p$ is a subrelation of $\prec_c$, $t_1 \prec_c t_3$ and $t_2 \prec_c t_3$. Thus by backward linearity of $\prec_c$, either $t_1 = t_2$ or $t_1 \prec_c t_2$ or $t_2 \prec_c t_1$. If $t_1 \prec_c t_2$ then (since $t_2 \prec_c t_3$ and $t_1 \prec_p t_3$) by (CP) $t_1 \prec_p t_2$. If $t_2 \prec_c t_1$ then (since $t_1 \prec_c t_3$ and $t_2 \prec_p t_3$) by (CP) $t_2 \prec_p t_1$. Next we show that (CP) implies (R.6). Let $t_1 \prec_p t_3$ and $t_2 \prec_c t_3$. Then, since $\prec_p$ is a subrelation of $\prec_c$, $t_1 \prec_c t_3$. It follows from backward linearity of $\prec_c$ that either (a) $t_1 = t_2$ or (b) $t_2 \prec_c t_1$ or (c) $t_1 \prec_c t_2$. In case (c) it follows from (CP) that $t_2 \prec_p t_3$.
Finally we show that the conjunction of (R.5) and (R.6) implies (CP). Let $t_1 \prec_c t_2$, $t_2 \prec_c t_3$ and $t_1 \prec_p t_3$. Since $t_1 \prec_c t_2$, by antisymmetry of $\prec_c$

$$t_1 \neq t_2 \quad and \quad t_2 \not\prec_c t_1. \tag{2.1}$$

Thus, by (R.6), $t_2 \prec_p t_3$. Hence, by (R.5) and (2.1) $t_1 \prec_p t_2$ (the case $t_2 \prec_p t_1$ is ruled out by (2.1) and the fact that $\prec_p$ is a subrelation of $\prec_c$). ■

Corollary 2.3. *In a tree-frame (CP′) is equivalent to (CP) which, in turn, is equivalent to the conjunction of (R.5) and (R.6).*

We view (CP) as a *minimum* requirement on the notion of prediction. The purpose of this paper is to investigate the modal logic of the most basic system. Depending on the application or interpretation one has in mind, further restrictions might be necessary. Some of them are discussed in Section 4. [6]

[6] It is worth noting that one property which we have *not* imposed is that the predicted future of t belong to the same future history, that is, we do not require forward linearity of $\prec_p$: if $t_1 \prec_p t_2$ and $t_1 \prec_p t_3$ then either $t_2 = t_3$ or $t_2 \prec_p t_3$ or $t_3 \prec_p t_2$. Indeed, although sometimes we make sharp predictions, often we do not.

We conclude this section by showing that (CP) incorporates a principle of "minimum revision of prediction".[7] Let $C(t) = \{t' \in T : t \prec_c t'\}$ be the set of $\prec_c$-successors of t (the conceivable future of t) and $P(t) = \{t' \in T : t \prec_p t'\}$ the set of $\prec_p$-successors of t (the predicted or likely future of t). The principle of minimum revision of prediction states that if t_2 is in the conceivable future of t_1, and the predicted future of t_1 has a non-empty intersection with the conceivable future of t_2, then the predicted future of t_2 should coincide with that intersection. Formally ('MR' stands for 'Minimum Revision'),

$$(MR) \qquad If \;\; t_1 \prec_c t_2 \;\; and \;\; P(t_1) \cap C(t_2) \neq \emptyset, \;\; then \;\; P(t_2) = P(t_1) \cap C(t_2).$$

Lemma 2.4. *If $\prec_p$ is transitive, (CP) implies (MR).*

Proof. Assume that $t_1 \prec_c t_2$ and $P(t_1) \cap C(t_2) \neq \emptyset$. First we show that $P(t_1) \cap C(t_2) \subseteq P(t_2)$. Let $t_3 \in P(t_1) \cap C(t_2)$, that is, $t_1 \prec_p t_3$ and $t_2 \prec_c t_3$. Then by (CP) $t_2 \prec_p t_3$, that is, $t_3 \in P(t_2)$. Next we show that $P(t_2) \subseteq P(t_1) \cap C(t_2)$. Fix an arbitrary $t_3 \in P(t_1) \cap C(t_2)$ (there exists one, because $P(t_1) \cap C(t_2) \neq \emptyset$). Then (since $t_1 \prec_c t_2$) by (CP) $t_1 \prec_p t_2$. Fix an arbitrary $t \in P(t_2)$, that is, $t_2 \prec_p t$. By transitivity of $\prec_p$, $t_1 \prec_p t$, that is, $t \in P(t_1)$. Hence $P(t_2) \subseteq P(t_1)$. Finally, since $\prec_p$ is a subrelation of $\prec_c$, $P(t_2) \subseteq C(t_2)$. ■

Thus in tree-frames (CP) implies (MR).[8]

3. Syntax

We consider a propositional language with four modal operators: G_c, H_c, G_p and H_p. The intended interpretation is as follows:

$G_c\phi$: "it is going to be the case in every *conceivable* future that ϕ"

$H_c\phi$: "it has always been the case that ϕ"

$G_p\phi$: "it is going to be the case in every *predicted* future that ϕ"

[7] In a different context (of knowledge and belief) this principle of minimum revision captures the qualitative part of Bayes' conditionalization rule and has been axiomatized in Battigalli and Bonanno (1997).

[8] On the other hand, (MR) does not imply (CP), as the following example shows: $T = \{t_1, t_2, t_3\}$, $C(t_1) = \{t_2, t_3\}$, $C(t_2) = P(t_1) = P(t_2) = \{t_3\}$, $C(t_3) = P(t_3) = \emptyset$. Here (MR) is satisfied but not (CP) since $t_2 \notin P(t_1)$.

$H_p\phi$: "it has always been the case at every past date at which today was predicted that ϕ".

The formal language is built in the familiar way from the following components: a countable set S of sentence letters (representing atomic propositions), the connectives $\neg$ and $\vee$ (from which the other connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual) and the modal operators G_c, H_c, G_p, H_p.[9] Let $F_c\phi \overset{def}{=} \neg G_c \neg\phi$, $P_c\phi \overset{def}{=} \neg H_c \neg\phi$, $F_p\phi \overset{def}{=} \neg G_p \neg\phi$ and $P_p\phi \overset{def}{=} \neg H_p \neg\phi$. Thus the intended interpretation is:

$F_c\phi$: "at some *conceivable* future date it will be the case that ϕ "

$P_c\phi$: "at some past date it was the case that ϕ "

$F_p\phi$: "at some *predicted* future date it will be the case that ϕ "

$P_p\phi$: "at some past date at which today was predicted it was the case that ϕ ".

Given a frame $\langle T, \prec_c, \prec_p \rangle$ one obtains a *model* $\mathcal{M}$ *based on it* by adding a function $V : S \rightarrow 2^T$ (where 2^T denotes the set of subsets of T) that associates with every sentence letter q the set of dates at which q is true. For every formula ϕ, the truth set of ϕ in $\mathcal{M}$, denoted by $\|\phi\|^{\mathcal{M}}$, is defined as usual: if $\phi = (q)$ where q is a sentence letter, then $\|\phi\|^{\mathcal{M}} = V(q)$, $\|\neg\phi\|^{\mathcal{M}} = T \setminus \|\phi\|^{\mathcal{M}}$, $\|\phi \vee \psi\|^{\mathcal{M}} = \|\phi\|^{\mathcal{M}} \cup \|\psi\|^{\mathcal{M}}$and

$$\|G_c\phi\|^{\mathcal{M}} = \left\{ t \in T : \forall t' \in T \text{ if } t \prec_c t' \text{ then } t' \in \|\phi\|^{\mathcal{M}} \right\}$$
$$\|H_c\phi\|^{\mathcal{M}} = \left\{ t \in T : \forall t'' \in T \text{ if } t'' \prec_c t \text{ then } t'' \in \|\phi\|^{\mathcal{M}} \right\}$$
$$\|G_p\phi\|^{\mathcal{M}} = \left\{ t \in T : \forall t' \in T \text{ if } t \prec_p t' \text{ then } t' \in \|\phi\|^{\mathcal{M}} \right\}$$

[9] The set Φ of formulae is thus obtained from the sentence letters by closing with respect to negation, disjunction and the operators G_c, H_c, G_p and H_p: (i) for every $q \in S$, $(q) \in \Phi$, (ii) if $\phi, \psi \in \Phi$ then all of the following belong to Φ: $(\neg\phi)$, $(\phi \vee \psi)$, $G_c\phi$, $H_c\phi$, $G_p\phi$, and $H_p\phi$.

$$\|H_p\phi\|^{\mathcal{M}} = \left\{t \in T : \forall t'' \in T \text{ if } t'' \prec_p t \text{ then } t'' \in \|\phi\|^{\mathcal{M}}\right\}.$$[10]

If $t \in \|\phi\|^{\mathcal{M}}$ we say that ϕ is *true at time* t *in model* $\mathcal{M}$. An alternative notation for $t \in \|\phi\|^{\mathcal{M}}$ is $\mathcal{M}, t \models \phi$ and an alternative notation for $t \notin \|\phi\|^{\mathcal{M}}$ is $\mathcal{M}, t \not\models \phi$. Thus $G_c\phi$ (resp. $G_p\phi$) is true at time t if ϕ is true at *every* conceivable (resp. predicted) future of t, while $F_c\phi$ (resp. $F_p\phi$) is true at time t if ϕ is true at *some* conceivable (resp. predicted) future of t. Similarly for $H_c\phi$, $H_p\phi$, $P_c\phi$ and $P_p\phi$.

A formula ϕ is *satisfiable* in a frame if there is a model $\mathcal{M}$ based on it and a time t such that $\mathcal{M}, t \models \phi$. A formula ϕ is *valid in model* $\mathcal{M}$ if $\|\phi\|^{\mathcal{M}} = T$, that is, if ϕ is true at every date $t \in T$. A formula ϕ is *valid in a frame* if it is valid in every model based on it.

Consider the following axiom schemata:

(A.1) $G_c\phi \to G_cG_c\phi$

(A.2) $P_c\phi \wedge P_c\psi \to P_c(\phi \wedge \psi) \vee P_c(\phi \wedge P_c\psi) \vee P_c(P_c\phi \wedge \psi)$

(A.3) $G_c\phi \to G_p\phi$

(A.4) $G_p\phi \to G_pG_p\phi$

(A.5) $P_p\phi \wedge P_p\psi \to P_p(\phi \wedge \psi) \vee P_p(\phi \wedge P_p\psi) \vee P_p(P_p\phi \wedge \psi)$

(A.6) $P_p\phi \wedge P_c\psi \to P_p(\phi \wedge \psi) \vee P_p(\phi \wedge P_c\psi) \vee P_p(P_c\phi \wedge \psi)$

The following characterization is straightforward.[11]

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[10] Thus

$\|F_c\phi\|^{\mathcal{M}} = \left\{t \in T : \exists t' \in T \text{ such that } t \prec_c t' \text{ and } t' \in \|\phi\|^{\mathcal{M}}\right\}$

$\|P_c\phi\|^{\mathcal{M}} = \left\{t \in T : \exists t'' \in T \text{ such that } t'' \prec_c t \text{ and } t'' \in \|\phi\|^{\mathcal{M}}\right\}$

$\|F_p\phi\|^{\mathcal{M}} = \left\{t \in T : \exists t' \in T \text{ such that } t \prec_p t' \text{ and } t' \in \|\phi\|^{\mathcal{M}}\right\}$

$\|P_p\phi\|^{\mathcal{M}} = \left\{t \in T : \exists t'' \in T \text{ such that } t'' \prec_p t \text{ and } t'' \in \|\phi\|^{\mathcal{M}}\right\}$

[11] For (a), (b), (c) and (e) see Burgess (1984). For (d) see van der Hoek (1983). To prove (f), consider a frame that satisfies (R.6) and any model based on it. Fix an arbitrary $t_3 \in T$ and arbitrary formulae ϕ and ψ. Suppose that $t_3 \models P_p\phi \wedge P_c\psi$. Then there exist t_1 and t_2 such that $t_1 \prec_p t_3$, $t_2 \prec_c t_3$, $t_1 \models \phi$ and $t_2 \models \psi$. By (R.6) either $t_1 = t_2$ (in which case $t_3 \models P_p(\phi \wedge \psi)$) or $t_2 \prec_c t_1$ (in which case $t_3 \models P_p(\phi \wedge P_c\psi)$) or $t_1 \prec_c t_2$ and $t_2 \prec_p t_3$ (in which case $t_3 \models P_p(P_c\phi \wedge \psi)$). Conversely, consider a frame that violates (R.6). Then there exist $t_1, t_2, t_3 \in T$ such that $t_1 \prec_p t_3$, $t_2 \prec_c t_3$, $t_1 \neq t_2$, $t_2 \not\prec_c t_1$ and if $t_1 \prec_c t_2$ then $t_2 \not\prec_p t_3$. Let q and s be atomic sentences and consider a model based on this frame where $\|q\| = \{t_1\}$

Lemma 3.1. *Let $\mathcal{F} = \langle T, \prec_c, \prec_p \rangle$ be an arbitrary frame (not necessarily a tree-frame). Then*
(a) $\mathcal{F}$ satisfies (R.1) if and only if (A.1) is valid in $\mathcal{F}$.
(b) $\mathcal{F}$ satisfies (R.2) if and only if (A.2) is valid in $\mathcal{F}$.
(c) $\mathcal{F}$ satisfies (R.3) if and only if (A.3) is valid in $\mathcal{F}$.
(d) $\mathcal{F}$ satisfies (R.4) if and only if (A.4) is valid in $\mathcal{F}$.
(e) $\mathcal{F}$ satisfies (R.5) if and only if (A.5) is valid in $\mathcal{F}$.
(f) $\mathcal{F}$ satisfies (R.6) if and only if (A.6) is valid in $\mathcal{F}$.

Definition 3.2. *A P-frame is a tree-frame that satisfies (R.5) and (R.6).*

We denote by $\mathbb{L}_0$ the basic system of temporal logic specified by the following axiom schemata and rules of inference.
Axiom schemata: all the classical tautologies as well as the following

(A.0a)	$G_c(\phi \to \psi) \to (G_c\phi \to G_c\psi)$	(A.0b)	$G_p(\phi \to \psi) \to (G_p\phi \to G_p\psi)$
(A.0c)	$H_c(\phi \to \psi) \to (H_c\phi \to H_c\psi)$	(A.0d)	$H_p(\phi \to \psi) \to (H_p\phi \to H_p\psi)$
(A.0e)	$\phi \to G_cP_c\phi$	(A.0f)	$\phi \to G_pP_p\phi$
(A.0g)	$\phi \to H_cF_c\phi$	(A.0h)	$\phi \to H_pF_p\phi$

Rules of inference:
Modus Ponens: from ϕ and $\phi \to \psi$ to infer ψ,
Temporal Generalization: from ϕ to infer $G_c\phi$, $H_c\phi$, $G_p\phi$, and $H_p\phi$.

Let $\mathbb{L}$ be the extension of $\mathbb{L}_0$ obtained by adding (A.1)-(A.6).

Theorem 3.3. *(SOUNDNESS THEOREM). $\mathbb{L}$ is sound with respect to the class of P-frames, that is, every theorem of $\mathbb{L}$ is valid in every P-frame.*

Proof. It suffices to show that each axiom is valid and that each rule of inference preserves validity. For a proof that the inference rules preserve validity and that axioms (A.0a)-(A.0h) are valid, see Burgess (1984). Validity of axioms (A.1)-(A.6) follows from Lemma 3.1. ■

and $\|s\| = \{t_2\}$. Then $t_3 \models P_pq \wedge P_cs$. Since $t_1 \neq t_2$, there is no t such that $t \models q \wedge s$. Thus $t_3 \nvDash P_p(q \wedge s)$. Since $t_2 \not\prec_c t_1$, there is no t such that $t \models q \wedge P_cs$. Thus $t_3 \nvDash P_p(q \wedge P_cs)$. Finally, since $t \models P_cq \wedge s$ if and only if $t = t_2$and $t_1 \prec_c t_2$ and since $t_1 \prec_c t_2$ implies $t_2 \not\prec_p t_3$, we have that $t_3 \nvDash P_p(P_cq \wedge s)$. Thus at t_3 the instance of (A.6) with $\phi = q$ and $\psi = s$ is false.

Theorem 3.4. *(COMPLETENESS THEOREM).* $\mathbb{L}$ *is complete with respect to the class of P-frames, that is, if a formula is valid in every P-frame then it is a theorem of* $\mathbb{L}$.

In order to prove the completeness theorem we need some preliminary results.[12] Let $Max\mathbb{L}$ be the set of maximal $\mathbb{L}$-consistent sets of formulae. Define the following relations on $Max\mathbb{L}$:

$$A \twoheadrightarrow_c B \text{ if and only if, for every formula } \phi \text{, if } G_c\phi \in A \text{ then } \phi \in B$$
$$A \twoheadrightarrow_p B \text{ if and only if, for every formula } \phi \text{, if } G_p\phi \in A \text{ then } \phi \in B$$

The next two lemmas are well-known (cf. Burgess 1984, Lemmas 1.6 and 1.7).

Lemma 3.5. *Let* $A, B \in Max\mathbb{L}$. *Then (1)-(4) below are equivalent and (i)-(iv) below are equivalent:*

(1) $A \twoheadrightarrow_c B$,
(2) for every formula ϕ, *if* $\phi \in A$ *then* $P_c\phi \in B$,
(3) for every formula ϕ, *if* $\phi \in B$ *then* $F_c\phi \in A$,
(4) for every formula ϕ, *if* $H_c\phi \in B$ *then* $\phi \in A$.

(i) $A \twoheadrightarrow_p B$,
(ii) for every formula ϕ, *if* $\phi \in A$ *then* $P_p\phi \in B$,
(iii) for every formula ϕ, *if* $\phi \in B$ *then* $F_p\phi \in A$,
(iv) for every formula ϕ, *if* $H_p\phi \in B$ *then* $\phi \in A$.

Lemma 3.6. *Let* $B \in Max\mathbb{L}$ *and* ϕ *be any formula. Then*

(a) if $F_c\phi \in B$, *then there exists a* $D \in Max\mathbb{L}$ *with* $B \twoheadrightarrow_c D$ *and* $\phi \in D$
(b) if $P_c\phi \in B$, *then there exists an* $A \in Max\mathbb{L}$ *with* $A \twoheadrightarrow_c B$ *and* $\phi \in A$
(c) if $F_p\phi \in B$, *then there exists a* $D \in Max\mathbb{L}$ *with* $B \twoheadrightarrow_p D$ *and* $\phi \in D$
(d) if $P_p\phi \in B$, *then there exists an* $A \in Max\mathbb{L}$ *with* $A \twoheadrightarrow_p B$ *and* $\phi \in A$.

Lemma 3.7. *The following holds.*

(a) By (A.1) the relation $\twoheadrightarrow_c$ *on* $Max\mathbb{L}$ *is transitive.*[13]
(b) By (A.2) the relation $\twoheadrightarrow_c$ *on* $Max\mathbb{L}$ *is backward linear.*[14]

[12] The completeness proof given below follows the approach put forward by Burgess (1984).

[13] More precisely, if $\mathbb{L}'$ is any extension of $\mathbb{L}_0$ that contains axiom (A.1), then the relation $\twoheadrightarrow_c$ on $Max\mathbb{L}'$ is transitive. A similar statement applies to (b)-(f).

[14] That is, if $A \twoheadrightarrow_c D$ and $B \twoheadrightarrow_c D$ then either $A = B$ or $A \twoheadrightarrow_c B$ or $B \twoheadrightarrow_c A$

(c) By (A.3) $\twoheadrightarrow_p$ *is a subrelation of* $\twoheadrightarrow_c$.[15]
(d) By (A.4) the relation $\twoheadrightarrow_p$ *on* $Max\mathbb{L}$ *is transitive.*
(e) By (A.5) the relation $\twoheadrightarrow_p$ *on* $Max\mathbb{L}$ *is backward linear.*[16]

(f) By (A.6) the relations $\twoheadrightarrow_c$ *and* $\twoheadrightarrow_p$ *satisfy the following property:*

$$\begin{array}{c} if\ \ A \twoheadrightarrow_p D\ \ and\ \ B \twoheadrightarrow_c D\ \ then\ \ either\ \ \ A = B\ \ or\ \ \ B \twoheadrightarrow_c A \\ or\ \ \ (A \twoheadrightarrow_c B\ \ \ and\ \ B \twoheadrightarrow_p D) \end{array}$$

Proof. For (a) and (d) see Burgess (1984, p. 103, first lemma). For (b) and (e) see Burgess (1984, p. 103, second lemma, adapting the proof given for forward linearity to the case of backward linearity). To prove (c), let $A \twoheadrightarrow_p B$ and let ϕ be an arbitrary formula and assume that $G_c\phi \in A$. We want to show that $\phi \in B$. By (A.3), $(G_c\phi \to G_p\phi) \in A$. Thus $G_p\phi \in A$. Hence by Lemma 3.5 (since $A \twoheadrightarrow_p B$) $\phi \in B$. We conclude by proving (f). Suppose that $A \twoheadrightarrow_p D$ and $B \twoheadrightarrow_c D$ and $A \neq B$ and $B \not\twoheadrightarrow_c A$ ($\not\twoheadrightarrow_c$ denotes the negation of $\twoheadrightarrow_c$). We need to show that ($A \twoheadrightarrow_c B$ and $B \twoheadrightarrow_p D$). We do this by showing that $A \not\twoheadrightarrow_c B$ yields a contradiction and $B \not\twoheadrightarrow_p D$ yields a contradiction. Since $A \neq B$, there exists a $\phi \in A$ such that $\neg\phi \in B$. Since $B \not\twoheadrightarrow_c A$, by Lemma 3.5 there exists a $\psi \in B$ such that $\neg P_c\psi \in A$. Thus

$$\phi \wedge \neg P_c\psi \in A \tag{3.1}$$

and

$$\psi \wedge \neg\phi \in B \tag{3.2}$$

Suppose that $A \not\twoheadrightarrow_c B$. Then by Lemma 3.5 there exists $\theta \in A$ such that $\neg P_c\theta \in B$. Then, using (3.1),

$$\theta \wedge \phi \wedge \neg P_c\psi \in A \tag{3.3}$$

and

$$\psi \wedge \neg\phi \wedge \neg P_c\theta \in B \tag{3.4}$$

By (3.3) and Lemma 3.5, since $A \twoheadrightarrow_p D$,

[15] That is, if $A \twoheadrightarrow_p B$ then $A \twoheadrightarrow_c B$
[16] That is, if $A \twoheadrightarrow_p D$ and $B \twoheadrightarrow_p D$ then either $A = B$ or $A \twoheadrightarrow_p B$ or $B \twoheadrightarrow_p A$.

$$P_p\left(\theta \wedge \phi \wedge \neg P_c\psi\right) \in D \tag{3.5}$$

By (3.4) and Lemma 3.5, since $B \twoheadrightarrow_c D$,

$$P_c\left(\psi \wedge \neg\phi \wedge \neg P_c\theta\right) \in D \tag{3.6}$$

Thus, by (3.5) and (3.6) and axiom (A.6), either

(i) $P_p(\theta \wedge \phi \wedge \neg P_c\psi \wedge \psi \wedge \neg\phi \wedge \neg P_c\theta) \in D$, or
(ii) $P_p\left(\theta \wedge \phi \wedge \neg P_c\psi \wedge P_c\left(\psi \wedge \neg\phi \wedge \neg P_c\theta\right)\right) \in D$, or
(iii) $P_p\left(P_c\left(\theta \wedge \phi \wedge \neg P_c\psi\right) \wedge \psi \wedge \neg\phi \wedge \neg P_c\theta\right) \in D$.

Case (i) is impossible because $\phi \wedge \neg\phi$ is a contradiction. Case (ii) is impossible because $P_c\left(\psi \wedge \neg\phi \wedge \neg P_c\theta\right)$ implies $P_c\psi$, contradicting $\neg P_c\psi$.[17] Case (iii) is impossible, since $P_c\left(\theta \wedge \phi \wedge \neg P_c\psi\right)$ implies $P_c\theta$, contradicting $\neg P_c\theta$. Hence it must be that $A \twoheadrightarrow_c B$. Suppose now that $B \not\twoheadrightarrow_p D$. Then by Lemma 3.5 there exists a $\gamma \in B$ such that

$$\neg P_p\gamma \in D \tag{3.7}$$

Then, using (3.2),

$$\gamma \wedge \psi \wedge \neg\phi \in B \tag{3.8}$$

Since $A \twoheadrightarrow_p D$, by (3.1) and Lemma 3.5,

$$P_p\left(\phi \wedge \neg P_c\psi\right) \in D \tag{3.9}$$

Since $B \twoheadrightarrow_c D$, by (3.8) and Lemma 3.5,

$$P_c(\gamma \wedge \psi \wedge \neg\phi) \in D \tag{3.10}$$

Thus by (3.9) and (3.10) and axiom (A.6)either

[17] More precisely, there would be a maximal consistend set E such that $E \twoheadrightarrow_p D$ and $\theta \wedge \phi \wedge \neg P_c\psi \wedge P_c\left(\psi \wedge \neg\phi \wedge \neg P_c\theta\right) \in E$. Then $\neg P_c\psi \in E$ and $P_c\left(\psi \wedge \neg\phi \wedge \neg P_c\theta\right) \in E$. Since $P_c\left(\psi \wedge \neg\phi \wedge \neg P_c\theta\right) \to P_c\psi$ is a theorem of $\mathbb{L}_0$, we would have that $P_c\psi \in E$. Thus $P_c\psi \wedge \neg P_c\psi \in E$, contradicting the definition of maximal consistent set.

(i) $P_p(\phi \wedge \neg P_c\psi \wedge \gamma \wedge \psi \wedge \neg\phi) \in D$, or
(ii) $P_p\left(\phi \wedge \neg P_c\psi \wedge P_c\left(\gamma \wedge \psi \wedge \neg\phi\right)\right) \in D$, or
(iii) $P_p\left(P_c\left(\phi \wedge \neg P_c\psi\right) \wedge \gamma \wedge \psi \wedge \neg\phi\right) \in D$.

Case (i) is impossible because $\phi \wedge \neg\phi$ is a contradiction. Case (ii) is impossible because $P_c\left(\gamma \wedge \psi \wedge \neg\phi\right)$ implies $P_c\psi$, contradicting $\neg P_c\psi$. Case (iii) is impossible because it implies $P_p\gamma \in D$, contradicting (3.7). Thus it must be $B \twoheadrightarrow_p D$. ■

Definition 3.8. *A partial canonical frame is a quadruple $\langle X, \prec_c, \prec_p, f\rangle$ such that: (1) X is a finite set, (2) $\langle X, \prec_c, \prec_p\rangle$ is a P-frame and (3) $f : X \to Max\mathbb{L}$. A partial canonical frame is* coherent *if, $\forall x, y \in X$,*
(1) if $x \prec_c y$ then $f(x) \twoheadrightarrow_c f(y)$, and
(2) if $x \prec_p y$ then $f(x) \twoheadrightarrow_p f(y)$.

Definition 3.9. *Let ϕ be a formula and $\langle X, \prec_c, \prec_p, f\rangle$ a partial canonical frame. We say that*
(a) $F_c\phi$ is not satisfied at $x \in X$, if $F_c\phi \in f(x)$ and there is no $y \in X$ such that $x \prec_c y$ and $\phi \in f(y)$.
(b) $F_p\phi$ is not satisfied at $x \in X$, if $F_p\phi \in f(x)$ and there is no $y \in X$ such that $x \prec_p y$ and $\phi \in f(y)$.
(c) $P_c\phi$ is not satisfied at $x \in X$, if $P_c\phi \in f(x)$ and there is no $y \in X$ such that $y \prec_c x$ and $\phi \in f(y)$.
(d) $P_p\phi$ is not satisfied at $x \in X$, if $P_p\phi \in f(x)$ and there is no $y \in X$ such that $y \prec_p x$ and $\phi \in f(y)$

Definition 3.10. *Given partial canonical frames $\langle X, \prec_c, \prec_p, f\rangle$ and $\left\langle X', \prec_c', \prec_p', f'\right\rangle$ we say that the latter is an* extension *of the former if (viewing relations and functions as sets of ordered pairs):*
(1) $X \subseteq X'$,
(2) $\prec_c \subseteq \prec_c'$,
(3) $\prec_p \subseteq \prec_p'$,
(4) $f \subseteq f'$.

Lemma 3.11. *EXTENSION LEMMA. Let $\langle X, \prec_c, \prec_p, f\rangle$ be a* coherent *partial canonical frame and let ϕ be a formula.*
***(a)** Suppose that $F_c\phi$ is not satisfied at $x \in X$. Then there exists a* coherent *extension $\left\langle X', \prec_c', \prec_p', f'\right\rangle$ of $\langle X, \prec_c, \prec_p, f\rangle$ and a $y \in X'$ such that $x \prec_c' y$ and $\phi \in f(y)$.*

(b) Suppose that $F_p\phi$ is not satisfied at $x \in X$. Then there exists a coherent extension $\left\langle X', \prec'_c, \prec'_p, f'\right\rangle$ and a $y \in X'$ such that $x \prec'_p y$ and $\phi \in f(y)$.
(c) Suppose that $P_c\phi$ is not satisfied at $x \in X$. Then there exists a coherent extension $\left\langle X', \prec'_c, \prec'_p, f'\right\rangle$ and a $y \in X'$ such that $y \prec'_c x$ and $\phi \in f(y)$.
(d) Suppose that $P_p\phi$ is not satisfied at $x \in X$. Then there exists a coherent extension $\left\langle X', \prec'_c, \prec'_p, f'\right\rangle$ and a $y \in X'$ such that $y \prec'_p x$ and $\phi \in f(y)$.

Proof. **(a)** Let $x \in X$ and $F_c\phi \in f(x)$. Then by Lemma 3.6 there is a $B \in Max\mathbb{L}$ such that $f(x) \twoheadrightarrow_c B$ and $\phi \in B$. Construct the following extension of $\langle X, \prec_c, \prec_p, f\rangle$ obtained by (i) adding a new point y, (ii) assigning the set B to y, and (iii) adding the pair (x, y) to $\prec_c$ (and any new pair needed to preserve transitivity), while no pairs are added to $\prec_p$. Let $y \in W \setminus X$ (where W is a proper superset of X, which we fix throughout this proof) and

$X' = X \cup \{y\}$
$\prec'_c = \prec_c \cup \{(x, y)\} \cup \{(v, y) : v \prec_c x\}$
$\prec'_p = \prec_p$
$f' = f \cup \{(y, B)\}$.

That the new partial frame is coherent follows from transitivity of $\twoheadrightarrow_c$ (cf. Lemma 3.7). It is also clear that the new frame is a P-frame, since the original frame was a P-frame, transitivity of $\prec_c$ has been preserved and no $\prec_p$-pairs have been added.

(b) Let $x \in X$ and $F_p\phi \in f(x)$. Then by Lemma 3.6 there is a $B \in Max\mathbb{L}$ such that $f(x) \twoheadrightarrow_p B$ and $\phi \in B$. Construct the following extension obtained by (i) adding a new point y, (ii) assigning the set B to y, and (iii) adding the pair (x, y) (and any new pair needed to preserve transitivity) to both $\prec_c$ and $\prec_p$. Let $y \in W \setminus X$ and

$X' = X \cup \{y\}$
$\prec'_c = \prec_c \cup \{(x, y)\} \cup \{(v, y) : v \prec_c x\}$
$\prec'_p = \prec_p \cup \{(x, y)\} \cup \{(v, y) : v \prec_p x\}$
$f' = f \cup \{(y, B)\}$.

That the new partial frame is coherent follows from Lemma 3.7 ($\twoheadrightarrow_p$ subrelation of $\twoheadrightarrow_c$, transitivity of $\twoheadrightarrow_c$ and $\twoheadrightarrow_p$). It is also clear that, given that the original frame was a P-frame, the new frame is also a P-frame: transitivity of both $\prec_c$ and $\prec_p$ has been preserved, every new $\prec_p$-pair is also a $\prec_c$-pair and for every $v \in X$, if $v \prec'_p y$ then the entire $\prec'_c$-path from v to y is also a $\prec'_p$-path (if $v = x$ this is true by construction; if $v \prec_c x$ then $v \prec'_p y$ requires $v \prec_p x$ hence, since the original frame is a P-frame, the entire $\prec_c$-path from v to x was also a $\prec_p$-path).

(c) Let $P_c\phi \in f(x)$ and suppose there is no $y \in X$ such that $y \prec_c x$ and $\phi \in f(y)$. We proceed by induction on the number n of $\prec_c$-predecessors of x in

X. Suppose $n = 0$. Since $P_c\phi \in f(x)$, by Lemma 3.6 there exists a $B \in Max\mathbb{L}$ such that $B \twoheadrightarrow_c f(x)$ and $\phi \in B$. Construct the following extension obtained by (i) adding a new point y, (ii) assigning the set B to y, and (iii) adding the pair (y, x) to $\prec_c$ (and any new pair needed to preserve transitivity), while no pairs are added to $\prec_p$. Let $y \in W \setminus X$ and

$X' = X \cup \{y\}$
$\prec'_c = \prec_c \cup \{(y, x)\} \cup \{(y, v) : x \prec_c v\}$
$\prec'_p = \prec_p$
$f' = f \cup \{(y, B)\}$.

Coherence follows from transitivity of $\twoheadrightarrow_c$. It is also clear that the new frame is a P-frame, since the original frame was a P-frame, transitivity of $\prec_c$ has been preserved and no $\prec_p$-pairs have been added.

Suppose now that $n \geq 1$. Let x' be the *immediate* $\prec_c$-predecessor of x in X (recall that X is finite). By our supposition, $\phi \notin f(x')$. If $P_c\phi \in f(x')$ then we can reduce (by appealing to transitivity of $\prec_c$ and $\twoheadrightarrow_c$) to the case $n-1$ by replacing x with x'. Assume therefore that $P_c\phi \notin f(x')$. Then, by definition of maximal consistent set, $(\neg\phi \wedge \neg P_c\phi) \in f(x')$. We need to distinguish two cases.

CASE 1: $x' \not\prec_p x$. By Lemma 3.6, since $P_c\phi \in f(x)$ there exists a $B \in Max\mathbb{L}$ such that $B \twoheadrightarrow_c f(x)$ and $\phi \in B$. Construct the following extension, obtained by (i) inserting a new point y between x' and x and assigning it the set B , (ii) adding the pairs (x', y) and (y, x) to $\prec_c$ (and any new pair needed to preserve transitivity), while no pairs are added to $\prec_p$. Let $y \in W \setminus X$ and

$X' = X \cup \{y\}$
$\prec'_c = \prec_c \cup \{(x', y), (y, x)\} \cup \{(v, y) : v \prec_c x'\} \cup \{(y, w) : x \prec_c w\}$
$\prec'_p = \prec_p$
$f' = f \cup \{(y, B)\}$.

To verify coherence, besides appealing to transitivity of $\twoheadrightarrow_c$, we need to show that $f(x') \twoheadrightarrow_c B$. By coherence of $\langle X, \prec_c, \prec_p, f\rangle$, $f(x') \twoheadrightarrow_c f(x)$. Thus, since $B \twoheadrightarrow_c f(x)$, by backward linearity of $\twoheadrightarrow_c$ (cf. Lemma 3.7) either (i) $f(x') = B$ or (ii) $B \twoheadrightarrow_c f(x')$ or (iii) $f(x') \twoheadrightarrow_c B$. Case (i) is ruled out by $\phi \in B$ and $\phi \notin f(x')$. Case (ii) is ruled out by $\phi \in B$ and $P_c\phi \notin f(x')$ (cf. Lemma 3.5). Thus $f(x') \twoheadrightarrow_c B$. Furthermore, it is clear that the new frame is a P-frame, since the original frame was a P-frame, transitivity of $\prec_c$ has been preserved and inserting a point between x' and x without adding any $\prec_p$-pairs would have violated property (CP) only if it had been the case that $x' \prec_p x$, contrary to our supposition.

CASE 2: $x' \prec_p x$. By coherence of $\langle X, \prec_c, \prec_p, f\rangle$, since $(\neg\phi \wedge \neg P_c\phi) \in f(x')$, $P_p(\neg\phi \wedge \neg P_c\phi) \in f(x)$ (cf. Lemma 3.5). Thus $P_p(\neg\phi \wedge \neg P_c\phi) \wedge P_c\phi \in f(x)$. By definition of maximal consistent set, axiom (A.6) belongs to $f(x)$. Thus

$$P_p(\neg\phi \wedge \neg P_c\phi \wedge \phi) \vee P_p(\neg\phi \wedge \neg P_c\phi \wedge P_c\phi) \vee P_p(P_c(\neg\phi \wedge \neg P_c\phi) \wedge \phi) \in f(x)$$

But $P_p(\neg\phi \wedge \neg P_c\phi \wedge \phi) \notin f(x)$ because $(\neg\phi \wedge \phi)$ is a contradiction. For the same reason, $P_p(\neg\phi \wedge \neg P_c\phi \wedge P_c\phi) \notin f(x)$. Thus $P_p(P_c(\neg\phi \wedge \neg P_c\phi) \wedge \phi) \in f(x)$. Then by Lemma 3.6, there exists a $D \in Max\mathbb{L}$ such that $D \twoheadrightarrow_p f(x)$ and $P_c(\neg\phi \wedge \neg P_c\phi) \wedge \phi \in D$. Construct the following extension, obtained by (i) inserting a new point y between x' and x and assigning it the set D, (ii) adding the pairs (x', y) and (y, x) (and any new pair needed to preserve transitivity) to both $\prec_c$ and $\prec_p$. Let $y \in W \setminus X$ and

$X' = X \cup \{y\}$

$\prec'_c = \prec_c \cup \{(x', y), (y, x)\} \cup \{(v, y) : v \prec_c x'\} \cup \{(y, w) : x \prec_c w\}$

$\prec'_p = \prec_p \cup \{(x', y), (y, x)\} \cup \{(v, y) : v \prec_p x'\} \cup \{(y, w) : x \prec_p w\}$

$f' = f \cup \{(y, D)\}$.

To verify coherence, besides appealing to transitivity of $\twoheadrightarrow_c$ and $\twoheadrightarrow_p$ and the fact that $\twoheadrightarrow_p$ is a subrelation of $\twoheadrightarrow_c$, we need to show that $f(x') \twoheadrightarrow_p D$. Since $f(x') \twoheadrightarrow_p f(x)$ (by coherence of $\langle X, \prec_c, \prec_p, f\rangle$) and $D \twoheadrightarrow_p f(x)$, by backward linearity of $\twoheadrightarrow_p$ (cf. Lemma 3.7) either (i) $f(x') = D$ or (ii) $D \twoheadrightarrow_p f(x')$ or (iii) $f(x') \twoheadrightarrow_p D$. Case (i) is ruled out by $\phi \in D$ and $\phi \notin f(x')$. Suppose (ii) were the case. Then, since $\phi \in D$, $P_p\phi \in f(x')$. But by (A.3) $P_p\phi \rightarrow P_c\phi \in f(x')$. Thus we would get $P_c\phi \in f(x')$, contradicting the fact that $\neg P_c\phi \in f(x')$. Hence it must be $f(x') \twoheadrightarrow_p D$. It is also clear that, given that the original frame was a P-frame, the new frame is also P-frame: transitivity of both $\prec_c$ and $\prec_p$ has been preserved, every new $\prec_p$-pair is also a $\prec_c$-pair and property (CP) is preserved since the new path from x' to x is both a $\prec_p$-path and a $\prec_c$-path.

(d) Let $P_p\phi \in f(x)$ and suppose there is no $y \in X$ such that $y \prec_p x$ and $\phi \in f(y)$. We need to distinguish two cases.

CASE 1: x has no $\prec_c$-predecessors (hence no $\prec_p$-predecessors) in X. Since $P_p\phi \in f(x)$, by Lemma 3.6 there exists a $B \in Max\mathbb{L}$ such that $B \twoheadrightarrow_p f(x)$ and $\phi \in B$. Construct the following extension obtained by (i) adding a new point y and assigning it the set B, (ii) adding the pair (y, x) (and any new pair needed to preserve transitivity) to both $\prec_c$ and $\prec_p$. Let $y \in W \setminus X$ and

$X' = X \cup \{y\}$

$\prec'_c = \prec_c \cup \{(y, x)\} \cup \{(y, v) : x \prec_c v\}$

$\prec'_p = \prec_p \cup \{(y, x)\} \cup \{(y, v) : x \prec_p v\}$

$f' = f \cup \{(y, B)\}$.

In this case coherence follows from transitivity of $\twoheadrightarrow_c$ and $\twoheadrightarrow_p$ and the fact that $\twoheadrightarrow_p$ is a subrelation of $\twoheadrightarrow_c$ (cf. Lemma 3.7). It is also clear that, given that the original frame was a P-frame, the new frame is also P-frame: transitivity of both $\prec_c$ and $\prec_p$ has been preserved, every new $\prec_p$-pair is also a $\prec_c$-pair and property (CP) is preserved since for every $w \in X$ if $y \prec'_p w$ then the entire $\prec'_c$-path from y to w is also a $\prec'_p$-path (if $w = x$ this is true by construction; if $x \prec_c w$ then $y \prec'_p w$ requires $x \prec_p w$ hence, since the original frame was a P-frame, the entire $\prec_c$-path from x to w was also a $\prec_p$-path).

CASE 2: x has a $\prec_c$-predecessor. Let x' be the *immediate* $\prec_c$-predecessor of x in X (recall that X is finite). We proceed by induction on the number n of $\prec_p$-predecessors of x in X. Suppose $n = 0$. Then $x' \not\prec_p x$. If $f(x') \twoheadrightarrow_p f(x)$ and $\phi \in f(x')$, then simply add $\{(x', x)\} \cup \{(x', v) : x \prec_p v\}$ to $\prec_p$. If $f(x') \twoheadrightarrow_p f(x)$ and $\phi \notin f(x')$ and $P_p\phi \in f(x')$, then add $\{(x', x)\} \cup \{(x', v) : x \prec_p v\}$ to $\prec_p$ and restart by replacing x with x'. If $f(x') \twoheadrightarrow_p f(x)$ and $\phi \notin f(x')$ and $P_p\phi \notin f(x')$, then add $\{(x', x)\} \cup \{(x', v) : x \prec_p v\}$ to $\prec_p$ and apply to x in the new frame the argument given below for the case where $n \geq 1$. Suppose therefore that $f(x') \not\twoheadrightarrow_p f(x)$. Since $P_p\phi \in f(x)$, by Lemma 3.6 there exists a $B \in Max\mathbb{L}$ such that

$$B \twoheadrightarrow_p f(x) \ \textit{and} \ \ \phi \in B. \tag{3.11}$$

Since $x' \prec_c x$, by coherence of $\langle X, \prec_c, \prec_p, f\rangle$, $f(x') \twoheadrightarrow_c f(x)$. Thus by (3.11) and (f) of Lemma 3.7, given our supposition that $f(x') \not\twoheadrightarrow_p f(x)$,

$$f(x') \twoheadrightarrow_c B. \tag{3.12}$$

Construct the following extension, obtained by (i) inserting a new point y between x and x' and assigning it the set B, (ii) adding the pairs (x', y) and (y, x) (and any new pairs needed to preserve transitivity) to $\prec_c$, and (iii) adding only the pair (y, x) (and any new pairs needed to preserve transitivity) to $\prec_p$. Let $y \in W \setminus X$ and

$X' = X \cup \{y\}$
$\prec'_c = \prec_c \cup \{(x', y), (y, x)\} \cup \{(v, y) : v \prec_c x'\} \cup \{(y, w) : x \prec_c w\}$
$\prec'_p = \prec_p \cup \{(y, x)\} \cup \{(y, v) : x \prec_p v\}$
$f' = f \cup \{(y, B)\}$.

Coherence follows from (3.11) and (3.12), the fact that $\twoheadrightarrow_p$ is a subrelation of $\twoheadrightarrow_c$ and transitivity of $\twoheadrightarrow_c$ and $\twoheadrightarrow_p$ (cf. Lemma 3.7). Furthermore, given that the original frame was a P-frame, the new frame is also a P-frame. In fact, transitivity of both $\prec_c$ and $\prec_p$ has been preserved and every new $\prec_p$-pair is also a $\prec_c$-pair. Moreover, given that we have not added (x', y) to $\prec_p$, (CP) would be violated only if x had a $\prec_p$-predecessor in the original frame, contrary to our supposition. Suppose now that $n \geq 1$ (recall that n is the number of $\prec_p$-predecessors of x in X). Let x' be the *immediate* $\prec_p$-predecessor of x in X (recall that X is finite). Since $P_p\phi$ is not satisfied at x, $\phi \notin f(x')$. If $P_p\phi \in f(x')$ then we can reduce (by appealing to transitivity of $\prec_p$ and $\twoheadrightarrow_p$) to the case $n-1$ by replacing x with x'. Assume therefore that $P_p\phi \notin f(x')$. Then, by definition of maximal consistent set, $(\neg\phi \wedge \neg P_p\phi) \in f(x')$. Thus, by coherence of $\langle X, \prec_c, \prec_p, f\rangle$, $P_p(\neg\phi \wedge \neg P_p\phi) \in f(x)$ (cf. Lemma 3.5). Hence

$$P_p\phi \wedge P_p(\neg\phi \wedge \neg P_p\phi) \in f(x)$$

By definition of maximal consistent set, axiom (A.5) belongs to $f(x)$. Thus,

$$P_p(\phi \wedge \neg\phi \wedge \neg P_p\phi) \vee P_p(P_p\phi \wedge \neg\phi \wedge \neg P_p\phi) \vee P_p(\phi \wedge P_p(\neg\phi \wedge \neg P_p\phi) \in f(x)$$

But $P_p(\phi \wedge \neg\phi \wedge \neg P_p\phi) \notin f(x)$ because $(\phi \wedge \neg\phi)$ is a contradiction. For the same reason, $P_p(P_p\phi \wedge \neg\phi \wedge \neg P_p\phi) \notin f(x)$. Thus $P_p(\phi \wedge P_p(\neg\phi \wedge \neg P_p\phi)) \in f(x)$. Then by Lemma 3.6, there exists a $D \in Max\mathbb{L}$ such that $D \twoheadrightarrow_p f(x)$ and $\phi \wedge P_p(\neg\phi \wedge \neg P_p\phi) \in D$. Construct the following extension, obtained by (i) inserting a new point y between x' and x and assigning it the set D and (ii) adding the pairs (x', y) and (y, x) (and any new pairs needed to preserve transitivity) to both $\prec_c$ and $\prec_p$. Let $y \in W \setminus X$ and

$X' = X \cup \{y\}$
$\prec'_c = \prec_c \cup \{(x', y), (y, x)\} \cup \{(v, y) : v \prec_c x'\} \cup \{(y, w) : x \prec_c w\}$
$\prec'_p = \prec_p \cup \{(x', y), (y, x)\} \cup \{(v, y) : v \prec_p x'\} \cup \{(y, w) : x \prec_p w\}$
$f' = f \cup \{(y, D)\}$.

To prove coherence, besides appealing to transitivity of $\twoheadrightarrow_c$ and $\twoheadrightarrow_p$ and the fact that $\twoheadrightarrow_p$ is a subrelation of $\twoheadrightarrow_c$, we have to show that $f(x') \twoheadrightarrow_p D$. Since $f(x') \twoheadrightarrow_p f(x)$ and $D \twoheadrightarrow_p f(x)$, by backward linearity of $\twoheadrightarrow_p$ (cf. Lemma 3.7) either (i) $f(x') = D$ or (ii) $D \twoheadrightarrow_p f(x')$ or (iii) $f(x') \twoheadrightarrow_p D$. Case (i) is ruled out by $\phi \in D$ and $\phi \notin f(x')$. Case (ii) is ruled out by $\phi \in D$ and $\neg P_p\phi \in f(x')$. Thus it must be $f(x') \twoheadrightarrow_p D$. Finally, the new frame is a P-frame since the original frame was a P-frame, all the new $\prec_p$-pairs are also $\prec_c$-pairs and property (CP) is preserved since the new path from x' to x is both a $\prec_p$-path and a $\prec_c$-path. ■

Definition 3.12. *A perfect chronicle on a P-frame $\langle T, \prec_c, \prec_p\rangle$ is a function $f : T \to Max\mathbb{L}$ such that $\langle T, \prec_c, \prec_p, f\rangle$ is coherent*[18] *and furthermore, $\forall t \in T$ and*

[18] That is, $t_1 \prec_c t_2$ implies $f(t_1) \twoheadrightarrow_c f(t_2)$ and $t_1 \prec_p t_2$ implies $f(t_1) \twoheadrightarrow_p f(t_2)$.

for every formula ϕ,

(a) if $F_c\phi \in f(t)$ then there exists a $t' \in T$ such that $t \prec_c t'$ and $\phi \in f(t')$,
(b) if $P_c\phi \in f(t)$ then there exists a $t'' \in T$ such that $t'' \prec_c t$ and $\phi \in f(t'')$,
(c) if $F_p\phi \in f(t)$ then there exists a $t' \in T$ such that $t \prec_p t'$ and $\phi \in f(t')$,
(d) if $P_p\phi \in f(t)$ then there exists a $t'' \in T$ such that $t'' \prec_p t$ and $\phi \in f(t'')$.

The following lemma is proved in Burgess (1984, Lemma 1.9).

Lemma 3.13. *If f is a perfect chronicle on $\langle T, \prec_c, \prec_p \rangle$ then any member of any $f(t)$ is satisfiable in $\langle T, \prec_c, \prec_p \rangle$.*

We can now prove the completeness theorem.

Proof. *(Completeness Theorem).* We have to show that if ϕ is an $\mathbb{L}$-consistent formula then it is satisfiable in a P-frame. By Lemma 3.13 it is sufficient to construct a perfect chronicle $\langle T, \prec_c, \prec_p, f\rangle$ such that $\phi \in f(t_0)$ for some t_0. Let W be a countably infinite set and let $(t_0, \phi_0), (t_1, \phi_1), (t_2, \phi_2), ...$ be an enumeration of $W \times \Phi$ (where Φ is the set of formulae). Construct the following coherent partial canonical frame: $T_0 = \{t_0\}$, $\prec_c^0 = \prec_p^0 = \emptyset$, $f(t_0) = A$, where A is a maximal $\mathbb{L}$-consistent set that contains ϕ. Let j_1 be the first integer such that $\langle t_{j_1}, \phi_{j_1} \rangle$ is not satisfied in $\langle T_0, \prec_c^0, \prec_p^0, f_0 \rangle$.[19] Apply the Extension Lemma to obtain a coherent partial canonical frame $\langle T_1, \prec_c^1, \prec_p^1, f_1 \rangle$ that satisfies $\langle t_{j_1}, \phi_{j_1} \rangle$. In general, let j_n be the first integer such that $\langle t_{j_n}, \phi_{j_n} \rangle$ is not satisfied in $\langle T_{n-1}, \prec_c^{n-1}, \prec_p^{n-1}, f_{n-1} \rangle$ and let $\langle T_n, \prec_c^n, \prec_p^n, f_n \rangle$ be the partial canonical frame obtained by applying the Extension Lemma to ϕ_{j_n} in $\langle T_{n-1}, \prec_c^{n-1}, \prec_p^{n-1}, f_{n-1} \rangle$. Let $\langle T, \prec_c, \prec_p, f\rangle$ be defined as follows:

$$T = \bigcup_{n=0}^{\infty} T_n, \quad \prec_c = \bigcup_{n=0}^{\infty} \prec_c^n, \quad \prec_p = \bigcup_{n=0}^{\infty} \prec_p^n, \quad f = \bigcup_{n=0}^{\infty} f_n$$

Then $\langle T, \prec_c, \prec_p, f\rangle$ is a perfect chronicle and $\phi \in f(t_0)$. ■

[19] That is, $t_{j_1} \in T_0$ and ϕ_{j_1} is equal to $F_c\psi$ or $F_p\psi$ or $P_c\psi$ or $P_p\psi$ for some formula ψ and ϕ_{j_1} is not satisfied at t_{j_1}.

4. Some theorems of $\mathbb{L}$ and possible extensions

In this section we highlight some aspects of system $\mathbb{L}$ and discuss possible extensions.

Consider the following axiom scheme:

$$(A.7) \qquad F_p P_c \phi \to (\phi \vee P_c \phi \vee F_p \phi)$$

Lemma 4.1. *Let $\mathcal{F}$ be an arbitrary frame (not necessarily a tree frame). Then (A.7) is valid in $\mathcal{F}$ if and only if $\mathcal{F}$ satisfies the following property: $\forall t_1, t_2, t_3 \in T$*

(R.7) if $t_1 \prec_p t_3$ and $t_2 \prec_c t_3$ then either (a) $t_1 = t_2$, or (b) $t_2 \prec_c t_1$ or (c) $t_1 \prec_p t_2$.

Proof. Let $\mathcal{F}$ be a frame that satisfies (R.7) and consider an arbitrary model based on it. Fix an arbitrary formula ϕ and an arbitrary date t_1. Suppose that $t_1 \models F_p P_c \phi$. Then there exist $t_2, t_3 \in T$ such that $t_1 \prec_p t_3$, $t_2 \prec_c t_3$ and $t_2 \models \phi$. By (R.7) either $t_1 = t_2$, in which case $t_1 \models \phi$, or $t_2 \prec_c t_1$, in which case $t_1 \models P_c \phi$, or $t_1 \prec_p t_2$, in which case $t_1 \models F_p \phi$. Thus in all three cases $t_1 \models \phi \vee P_c \phi \vee F_p \phi$. Conversely, let $\mathcal{F}$ be a frame that does not satisfy (R.7). Then there exist $t_1, t_2, t_3 \in T$ such that $t_1 \prec_p t_3$, $t_2 \prec_c t_3$, $t_1 \neq t_2$, $t_2 \not\prec_c t_1$ and $t_1 \not\prec_p t_2$. Let q be a sentence letter and consider a model based on $\mathcal{F}$ where $\|q\| = \{t_2\}$. Then all of the following are true at t_1: $F_p P_c q$, $\neg q$ (because $t_1 \neq t_2$), $\neg P_c q$ (because $t_2 \not\prec_c t_1$) and $\neg F_p q$ (because $t_1 \not\prec_p t_2$). Thus the following instance of (A.7) is false at t_1: $F_p P_c q \to (q \vee P_c q \vee F_p q)$. ■

Corollary 4.2. *(A.7) is a theorem of $\mathbb{L}$.*

Proof. First we show that if $\prec_p$ is a subrelation of $\prec_c$ and $\prec_c$ is backward linear, then (CP) implies (R.7). Let $t_1 \prec_p t_3$ and $t_2 \prec_c t_3$. Since $\prec_p$ is a subrelation of $\prec_c$, $t_1 \prec_c t_3$. Hence, by backward linearity of $\prec_c$, either (a) $t_1 = t_2$, or (b) $t_2 \prec_c t_1$ or (c) $t_1 \prec_c t_2$. In case (c), it follows from (CP) that $t_1 \prec_p t_2$. Thus every P-frame satisfies (R.7). Hence, by Lemma 4.1, (A.7) is valid in every P-frame. Thus by the completeness theorem, (A.7) is a theorem of $\mathbb{L}$. ■

In some applications it may make sense to require that, for every date t, the predicted future of t be non-empty, unless t is a terminal date (i.e. it has no $\prec_c$-successors). Semantically this property can be expressed as the requirement that $\prec_p$ be serial whenever $\prec_c$ is serial. The following lemma gives the axiom scheme that characterizes this property.

Lemma 4.3. *Let $\mathcal{F}$ be an arbitrary frame (not necessarily a tree frame). Then the following axiom scheme*

$$\text{(A.8)} \quad G_p\phi \wedge F_c\phi \rightarrow F_p\phi$$

is valid in $\mathcal{F}$ if and only if $\mathcal{F}$ satisfies the property that $\prec_p$ is serial whenever $\prec_c$ is serial, that is,

$$\text{(R.8)} \quad \forall t \in T, \text{ if } \exists t_1 \text{ such that } t \prec_c t_1, \text{ then } \exists t_2 \text{ such that } t \prec_p t_2.$$

Proof. Fix an arbitrary model based on a frame that satisfies the property that $\prec_p$ is serial whenever $\prec_c$ is serial. Fix an arbitrary $t \in T$ and an arbitrary formula ϕ. Suppose that $t \models G_p\phi \wedge F_c\phi$. Since $t \models F_c\phi$, $\exists t_1$ s.t. $t \prec_c t_1$ and $t_1 \models \phi$. Thus $\prec_c$ is serial at t. By the assumed property, $\prec_p$ is serial at t, that is, $\exists t_2$ s.t. $t \prec_p t_2$. Hence, since $t \models G_p\phi$, $t_2 \models \phi$. Thus $t \models F_p\phi$. Conversely, fix a frame that does not satisfy the above property. Then there exist $t, t_1 \in T$ s.t. $t \prec_c t_1$ and, $\forall t' \in T$, $t \not\prec_p t'$. Consider a model based on this frame where, for some atomic sentence q, $\|q\| = \{t_1\}$. Then $t \models F_cq$ and, since t has no $\prec_p$-successors, $t \models G_pq \wedge \neg F_pq$. Thus $G_pq \wedge F_cq \rightarrow F_pq$ is false at t. ■

Another possible extension of $\mathbb{L}$ can be obtained by adding the requirement that predictions be "unique", in the sense that the predicted future of any date t consist of points on the same branch out of t. This requirement is captured by the property of forward linearity of $\prec_p$, whose characterizing axiom is given in the following lemma (for a proof see Burgess, 1984).

Lemma 4.4. *Let $\mathcal{F}$ be an arbitrary frame (not necessarily a tree frame). Then the following axiom scheme*

$$\text{(A.9)} \quad F_p\phi \wedge F_p\psi \rightarrow F_p(\phi \wedge \psi) \vee F_p(\phi \wedge F_p\psi) \vee F_p(F_p\phi \wedge \psi)$$

is valid in $\mathcal{F}$ if and only if $\mathcal{F}$ satisfies the property of forward linearity of $\prec_p$:

$$\text{(R.9)} \quad \forall t_1, t_2, t_3 \in T, \text{ if } t_1 \prec_p t_2 \text{ and } t_1 \prec_p t_3, \text{ then either } t_2 = t_3, \text{ or } t_2 \prec_p t_3 \text{ or } t_3 \prec_p t_2.$$

It can be shown that the system obtained by adding (A.8) (resp. (A.9)) to $\mathbb{L}$ is sound and complete with respect to the class of P-frames that satisfy property (R.8) (resp. (R.9)). Similarly, adding both (A.8) and (A.9) to $\mathbb{L}$ yields a system that is sound and complete with respect to the class of P-frames that satisfy both (R.8) and (R.9).

5. Concluding remarks

The purpose of this paper was to isolate a minimal logic of prediction. Depending on the application and/or interpretation one has in mind, it may be desirable to extend the logic by adding further axioms. One such axiom, which was discussed in the previous section, is that "prediction spreads forward", in the sense that $\prec_p$ is serial whenever $\prec_c$ is serial (cf. axiom (A.8)). This property, together with (CP) and transitivity of $\prec_p$, implies that the predicted future of any moment t consists of a set of *branches* (i.e. maximal $\prec_c$-chains) through t. In this case, instead of the Kripke-style semantics used in this paper, an alternative approach is possible, namely the "Ockhamist" approach in which the truth of a formula is not evaluated at a single point in time, but at a pair consisting of a time point and a branch through it; the future operator F then refers to time points in this branch only. It follows that the evaluation of temporal formulae obeys the laws of linear time: the environment in which any such formula is evaluated is a single branch (or history). Because of this, in order to express the manifoldness of the future, a possibility operator $\Diamond$ is needed to enable one to quantify over the branches through a given moment (cf. Thomason, 1984 and Zanardo, 1996). Extending the analysis of this paper to the Ockhamist approach would require introducing two possibility operators, $\Diamond_c$ and $\Diamond_p$, with the following truth conditions within each model (t denotes a time point and b a branch through t): $(t, b) \models \Diamond_c \phi$ if there is a *conceivable* branch b' through t such that $(t, b') \models \phi$, and $(t, b) \models \Diamond_p \phi$ if there is a *predicted* branch b' through t such that $(t, b') \models \phi$.

Although the relation $\prec_p$ was interpreted in this paper as expressing the notion of prediction, other interpretations are possible. For instance, if the logic $\mathbb{L}$ is augmented with axioms (A.8) and (A.9), then one obtains a system where for very instant t the set $\{t' : t \prec_p t'\}$ gives a unique history after t: such a history could be interpreted as the *actual future* of t. Thus this logic could be used as an axiomatization of the point of view called Actualism: actualists hold that there is a particular possible future of a given moment t, i.e. the actual future of t, to which the future operator refers. In the logic $\mathbb{L}$ augmented with (A.8) and (A.9), one would have two future operators: F_p and F_c. The interpretation of $F_p \phi$ would be "it will be the case that ϕ in the actual future" while the interpretation of $F_c \phi$ would be "there is a conceivable future time at which ϕ will be true, although not necessarily in the actual future".

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