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# THE PROBABILITY APPROACH TO GENERAL EQUILIBRIUM WITH PRODUCTION

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**Abstract:** We develop an alternative approach to the general equilibrium analysis of a stochastic production economy when firms' choices of investment influence the probability distributions of their output. Using a normative approach we derive the criterion that a firm should maximize to obtain a Pareto optimal equilibrium: the criterion expresses the firm's contribution to the expected social utility of output, and is not the linear criterion of market value. If firms do not know agents utility functions, and are restricted to using the information conveyed by prices then they can construct an approximate criterion which leads to a second-best choice of investment which, in examples, is found to be close to the first best.

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## 1. Introduction

Just as there are two ways of analyzing a random variable, so there are two approaches to modeling a production economy under uncertainty. The first approach introduces a set of states of nature with fixed probabilities of occurrence and lets firms' actions influence the quantities of the goods produced in each state: this is the approach introduced by Arrow and Debreu (1953, 1959), which constitutes the reference model of general equilibrium. The second approach introduces a probability distribution over possible outcomes, and lets firms' actions influence the probabilities of the outcomes: this approach has not been systematically explored in general equilibrium[1] and is the focus of this paper. While the first approach is analogous to modeling a random variable as a map from a state space to the real line, the second studies a random variable through the probability distribution it induces on the outcome (range) space. Since the real-world financial contracts which share the risks and direct investment activity of firms are typically based on outcomes and not on primitive states of nature, we argue that the latter approach is a natural candidate for a general equilibrium analysis of a production economy under uncertainty.

As far as the description of production possibilities is concerned, the state-space representation is more general. Given a probability representation in which investment affects the probabilities of the outcomes, there exists a state-space representation with fixed probabilities for the states in which investment influences the quantity of output in each state, and in which the induced probability distribution on outcomes coincide with the probability representation.[2] A simple example suffices to illustrate the construction.[3] Consider a firm with two possible outputs $y_L < y_H$ and two possible investment levels $a_L < a_H$. The probability approach models the probability $p$ of the high outcome as a function of investment, for example $p(a_L) = \frac{1}{4}$ and $p(a_H) = \frac{1}{2}$, reflecting the fact that investment in higher grade personnel or equipment makes the probability of the high output $y_H$ more likely. The same production possibilities can be described by a model with four states

[1] There is a general equilibrium literature with moral hazard which uses the probability approach (Prescott-Townsend (1984 a,b), Kocherlakota (1998), Bisin-Gottardi (1999), Lisboa (2001)), Zame (2006)). Since in these papers it is assumed that there is a continuum of agents or firms of each type who are subject to independent shocks, probabilities become proportions and uncertainty in essence disappears. Thus the issues related to risk aversion and aggregate uncertainty, which we study and which arise when there are finitely many agents and firms, are not studied in these papers.

[2] Roughly speaking this is a modified version of Kolmogorov's extension theorem which states that given a probability representation of a random variable by a distribution function $F : \mathbb{R} \to [0, 1]$, there exists a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ such that the random variable can be viewed as a map from $\Omega$ to $\mathbb{R}$ and the probability distribution induced by $\mathcal{P}$ coincides wih $F$.

[3] We thank a referee for suggesting this example as a simple way of showing the relation between the state-of-nature and the probability representation.

of nature and a production function $f(a_L) = (y_L, y_L, y_L, y_H)$, $f(a_H) = (y_L, y_L, y_H, y_H)$ in which the probability of each state is $\frac{1}{4}$ independent of the firm's investment: investment now affects the quantity produced in each state. In both models the probability of a high outcome is $\frac{1}{4}$ if investment is low and $\frac{1}{2}$ if investment is high, and the two models are equivalent for an investor with expected utility preferences.

If description of production possibilities were the only criterion for the choice of a model, then the choice would be clear: the state-of-nature model, being more general, should be the reference model. However the description of production possibilities is only half the model. The other half describes the contracts (markets) which are used to share risks and direct investment. A state-of-nature model assumes that contracts are contingent on the exogenous states of nature, and in a model with production, essentially the only case which yields a well-defined objective for a firm is when markets are complete with respect to the states of nature. In the probability model, states of nature are left unspecified and the contracts are assumed to be contingent on the possible outcomes of the firms' investment. Implicit in this approach is the assumption that even if in principle with "sufficient knowledge" the outcome of each firm's investment could be traced back to primitive causes—states of nature whose probability of occurrence is independent of firms' actions—these states are too difficult to describe and/or to verify by third parties to permit contracts based on their occurrence to be traded.[4] That this assumption is realistic seems to be confirmed by the striking fact that the contracts which are used to finance investment and share production risks—bonds, equity and derivative securities—are either non contingent or based on realized profits and prices, rather than on exogenous events with fixed probabilities.

These security markets have undergone a remarkable development in the last thirty years with the introduction of more and more derivative contracts. We will use this observation to justify our assumption that the markets are sufficiently rich to span the uncertainty in the outcomes of the firms: this means that it is possible (at a cost) to find a portfolio of bonds, equity contracts and derivatives whose payoff is one unit if a given outcome for the firms is realized, and nothing otherwise. As Ross (1976) showed, in a two-period model this is always possible if a sufficient number of options are introduced. In this paper we assume that this *full spanning* assumption is

[4] The difficulty of using the state-of-nature approach has mainly been discussed in the context of insurance (Ehrlich-Becker (1973), Marshall (1976)). Marshall argues that the typical reason that the approach cannot be used is because it would be "too costly" for insurance companies to specify precisely in a contract ex-ante, and to verify ex-post, the states of nature that can lead to an accident and whose probabilities are independent of the actions of the insured agent. Actual insurance contracts are written directly on the "value of the loss", an economic outcome which is typically easy to describe and verify, and whose probability of occurrence is almost certain to be influenced by the actions of the insured agent.

satisfied. In the formal model the full spanning assumption appears as the assumption that for each possible outcome there is a contract which delivers one unit of income if this outcome is realized and this contract has a well defined price at the initial date.

The full spanning assumption in a probability representation of a production economy is typically much less restrictive than the assumption of complete markets in the associated state-space representation. If contracts are based on outcomes and there is full spanning, then the number of independent securities is equal to the number of outcomes. If the probabilities of outcomes are influenced by investment, but the probabilities of the states are to remain independent of investment, then any state-space representation must have more states than outcomes, so that markets are necessarily incomplete in the sense of GEI. In the simple example given above, although it is convenient to choose a representation with four equiprobable states, we could make do with three states, but no less: in order that the probability of the high output is different when the investment levels differ, there must be at least one state in which high investment results in the high outcome and low investment results in the low outcome. Thus there must be more states than outcomes and markets are incomplete.

Since there is no satisfactory resolution of the choice of the objective function for firms when markets are incomplete, if we take as a stylized fact that contracts depend on outcomes, then there is no point in adopting a state-space representation of a production economy. It is better to proceed directly to a new analysis of equilibrium using the probability representation.

We consider therefore a simple two-period model of a production economy in which firms make investment decisions at date 0 which influence the probability distribution of their output at date 1. If we anticipate that under favorable conditions an equilibrium will be Pareto optimal, then the first task is to derive what firms "should do", namely the criterion that they should adopt to lead the economy to Pareto optimality : we can then discuss whether firms will have an incentive to adopt such a criterion. The first-order conditions for Pareto optimality lead to a nonlinear criterion which expresses a firm's contribution to the social utility of date 1 output, net of the cost of investment at date 0.

Let us try to explain in an intuitive way why such a criterion emerges when we use the probability approach, and why it differs from the standard market-value criterion when the state space representation is used. Note first that when the distribution of aggregate output among consumers is efficient and agents have Von-Neumann-Morgernstern preferences, the social utility of date 1 consumption (output) is of the form $\sum_{s\in S} p_s\Phi(Y_s)$ where $(p_s)_{s\in S}$ are the probabilities, $(Y_s)_{s\in S}$ is aggregate output and $\Phi$ is a social utility function. If we adopt a state-space representation, then

$S$ is the set of states of nature, for each $s \in S$ the probability $p_s$ is fixed, and firms' investments influence the quantity of output in each state: the marginal social benefit of increasing firm $k$'s investment $a_k$ is $\sum_{s\in S} p_s \Phi'(Y_s)\frac{\partial Y_s}{\partial a_k}$. Since $\frac{\partial Y_s}{\partial a_k} = \frac{\partial y^k_s}{\partial a_k}$, where $y^k$ is the output of firm $k$, and since the prices are the marginal social utilities $(p_s\Phi'(Y_s))_{s\in S}$, the marginal revenue of firm $k$ coincides with the marginal social benefit of investment. Thus when markets are complete maximizing the present value of profit (the market value of the firm) leads to an efficient choice of investment. If on the other hand we adopt the probability representation, then $S$ denotes the the set of possible date 1 outcomes of the firms, and the firms' investment decisions $a$ influence the probabilities $p_s(a)$: the marginal social benefit of increasing firm $k$'s investment is then $\sum_{s\in S} \frac{\partial p_s}{\partial a_k}\Phi(Y_s)$. Thus with the probability representation it is social utility rather than marginal utility which defines a firm's investment criterion.

Applying a normative approach to the probability representation of production thus leads to a concept of equilibrium, which we call a *strong firm-expected-utility* (SFEU) *equilibrium*, in which investors (consumers) share risks on markets, and firms choose investment to maximize the expected social utility criterion. The qualifier "strong" refers to the strong informational assumption required to implement such an equilibrium: to know the social utility function, firms must know all the individual agents' utility functions—in particular their risk aversion. So at first sight with the probability representation, prices seem to have lost their fundamental role of conveying all the requisite information to firms. However all is not lost. For what firms need to know to make socially efficient investment decisions are differences in social utility associated with different outcomes, and this, or at least an estimate of it, can be obtained by "integrating" the marginal utilities whose values are given by the prices. Thus in the end firms can back out an estimate of the social utility function from the prices, so that prices once again convey the requisite information to firms.

To formalize this informational role of prices we introduce a new concept of constrained optimality: this is a Pareto optimum constrained by the condition that the "planner" does not have access to more information regarding agents' utility functions than that conveyed by the prices, and we call this a *second-best optimum*. The concept of equilibrium which satisfies the first-order conditions for second-best optimality is then called an FEU equilibrium. In such an equilibrium each firm constructs an estimate of the social utility function using the information on marginal utilities contained in the prices, and then chooses its investment to maximize its contribution to expected social welfare using this estimated social utility function.

As Arrow (1983) pointed out, when a stochastic production model departs from the state-of-nature representation, lack of convexity may present problems. In Section 6 we examine the convexity (concavity) assumptions needed to obtained existence and (constrained) optimality of equilibrium. Unlike in the state-of-nature model, with the probability model the convexity assumptions needed to prove existence of an equilibrium are weaker than those needed to obtain optimality. To get existence, it suffices to have a stochastic version of decreasing returns to scale for the investment of each firm. This however is not sufficient to imply the joint concavity assumption on the upper-cumulative distribution function of aggregate output needed to prove optimality of equilibrium.

The paper is organized as follows. Section 2 presents the model of a production economy using the probability approach. Section 3 contains the first-best analysis which leads to the expected utility criterion for each firm and the associated concept of a strong FEU equilibrium. Section 4 introduces the concept of second-best optimality where agents' utility functions are not known, and Section 5 studies the related weaker concept of an FEU equilibrium in which firms only need to know prices. Section 6 establishes the normative properties of an FEU and a strong FEU equilibrium, and gives conditions under which an equilibrium exists. Section 7 concludes with some remarks on directions for future research.

## 2. Probability Approach to Production Economy

This section presents the basic model of a production economy using the probability approach. To contrast the properties of this model, in which actions influence probabilities of outcomes with the properties of the standard state-of-nature production model, we focus on the simplest model of a two-period finance economy in which agents have separable Von-Neumann-Morgenstern utilities and the only risks to which they are exposed are those which come from the production sector.

Consider therefore a two-period economy ($t = 0, 1$), with a single good (income) and a finite number of agents ($i = 1, \ldots, I$) and firms ($k = 1 \ldots, K$). Each firm makes an investment at date 0 which leads to a probability distribution over a finite number of possible outcomes (output levels) at date 1. Let $a_k \in \mathbb{R}_+$ denote the investment (action) of firm $k$ at date 0 and let $y^k$ denote the date 1 random output which can take the $S_k$ values $(y_1^k, \ldots, y_{S_k}^k)$, ranked in increasing order. With a slight abuse of notation we let $S_k$ denote the index set for the possible output levels of firm $k$ as well as the number of its elements.[5] A typical element of $S_k$ is denoted by $s_k$, and $s_k > s'_k$ implies $y_{s_k}^k > y_{s'_k}^k$. The outcome space for the economy is $S = S_1 \times \ldots S_K$ which describes all

[5]In this paper we use the same capital letter for a set and the number of its elements: thus $I$ is the number as well as the set of agents, $K$ is the number as well as the set of firms, ....

the possible outcomes for the $K$ firms of the economy. Thus if $s = (s_1, \ldots, s_K)$ is an element of $S$, then the associated vector of outputs of the $K$ firms is $y_s = (y^1_{s_1}, \ldots, y^K_{s_K})$. Using standard notation for a vector of random variables, let $y = (y_s)_{s\in S}$ denote the finite collection of possible outputs for the firms at date 1 and let $Y = \sum_{k\in K} y^k$ denote the associated aggregate output. For $s \in S$, let $p_s(a)$ denote the probability of outcome $s$ when the investment levels of the firms are $a = (a_1, \ldots, a_K) \in \mathbb{R}^K_+$: we assume $p_s(a) > 0$ for all $s \in S$ and $a \geq 0$.

**Assumption FS (full support):** *The function $p(a) = (p_s(a))_{s\in S}$ is differentiable*[6] *on $\mathbb{R}^K_+$ and for each investment level $a \in \mathbb{R}^K_+$, the support of $p(a) = (p_s(a))_{s\in S}$ is equal to $S$.*

Assumption (FS) implies that all outcomes $(y^k_{s_k})_{s_k\in S_k}$ of firm $k$ are possible for any level $a_k$ of investment: this may seem restrictive since it excludes the case of certainty where the output of a firm is a deterministic function of its input, or the case where only some of the values $(y^k_{s_k})_{s_k\in S_k}$ are possible with a certain level of investment. These cases can however be approximated by placing positive but very small probability on the appropriate part of the fixed support $S$. When we analyze the investment decision of a particular firm $k$, it is often convenient to write the outcome $s$ as $s = (s_k, s^{-k})$ where $s^{-k} = (s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_K)$, and use the same convention for the firms' investment decisions $a = (a_k, a^{-k})$.

Each agent $i \in I$ has initial resources consisting of an amount $w^i_0$ of income at date 0, and initial ownership shares $\delta^i = (\delta^i_k)_{k\in K}$ of the firms: agents have no initial endowment of income at date 1, so that all consumption at date 1 comes from the firms' outputs.

**Assumption IN (initial endowments):** *For each $i \in I$, agent $i$'s endowment is $(w^i_0, \delta^i) \in \mathbb{R}_+ \times \mathbb{R}^K_+$, and $\sum_{i\in I} w^i_0 > 0$, $\sum_{i\in I} \delta^i_k = 1$, for all $k \in K$.*

Assumption (IN) implies that agents have no idiosyncratic risks and that all the risks in the economy are production risks: agents' consumption streams will thus only vary with the outputs of the firms. Let $x^i = (x^i_0, (x^i_s)_{s\in S})$ denote a consumption stream for agent $i$. We assume that agent $i$'s preferences, represented by the utility function $U^i$, are separable across time and have the expected utility form for future risky consumption. To avoid boundary solutions which are not natural in a one-good (income) model, we assume that the marginal utility of consumption tends to infinity when consumption tends to zero: we say that a function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies the Inada condition if $\lim f(x) = +\infty$ when $x \to 0^+$.

[6]For brevity, we use the convention that "differentiable" means "continuously differentiable".

**Assumption EU (expected utility):** *For each $i \in I$, there exist increasing, differentiable, strictly concave functions $(u_0^i, u_1^i)$: $\mathbb{R}_+ \to \mathbb{R}$, satisfying the Inada condition, such that*

$$U^i(x^i; a) = u_0^i(x_0^i) + \sum_{s \in S} p_s(a) u_1^i(x_s^i)$$

An agent's utility $U^i(x^i; a)$ depends not only on the consumption stream $x^i$ but also on the probability $p(a)$ of the outcomes, which is determined by the investments $a$ made by the firms at date 0. It is convenient to let $\mathcal{E}(U, w_0, \delta, y, p)$ summarize the above economy, in which the agents' utility functions are $U = (U^i)_{i \in I}$, their endowments are $w_0 = (w_0^i)_{i \in I}$ and $\delta = (\delta^i)_{i \in I}$, and the production possibilities are represented by the firms' outcomes $y$ and the probability function $p$.

## 3. First-Best Analysis and Strong FEU Equilibrium

In this section we show how the first-order conditions for Pareto optimality lead to the first concept of equilibrium for the probability model of a production economy. Let $Y_s = \sum_{k \in K} y_{s_k}^k$ denote the aggregate output of the firms in outcome $s$, $s \in S$. An allocation $(a, x)$ is *feasible* if

$$\sum_{i \in I} x_0^i + \sum_{k \in K} a_k = \sum_{i \in I} w_0^i, \qquad \sum_{i \in I} x_s^i = Y_s, \; s \in S \tag{1}$$

An allocation $(\bar{a}, \bar{x})$ is *Pareto optimal* if and only if, for some weights $\mu = (\mu_i)_{i \in I} \in \mathbb{R}_+^I \setminus 0$, it is a solution to the problem of maximizing social welfare

$$\max_{a \in \mathbb{R}_+^K, \; x \in \mathbb{R}_+^{(S+1)I}} \sum_{i \in I} \mu_i \left( u_0^i(x_0^i) + \sum_{s \in S} p_s(a) u_1^i(x_s^i) \right) \tag{2}$$

subject to the feasibility constraints (1).

If $(\bar{\lambda}_0, (\bar{\lambda}_s)_{s \in S})$ are the Lagrange multipliers associated with the feasibility constraints, the FOCs for an interior solution to the maximization problem (2) are

$$\mu_i u_0^{i\prime}(\bar{x}_0^i) = \bar{\lambda}_0, \qquad \mu_i p_s(\bar{a}) u_1^{i\prime}(\bar{x}_s^i) = \bar{\lambda}_s, \; s \in S, \; i \in I \tag{3}$$

$$\sum_{s \in S} \frac{\partial p_s}{\partial a_k}(\bar{a}) \sum_{i \in I} \mu_i u_1^i(\bar{x}_s^i) = \bar{\lambda}_0, \; k \in K \tag{4}$$

The FOCs (3) express the equalization of the marginal rates of substitution of the agents required for an efficient distribution of output among agents: these conditions will be satisfied if markets are used to allocate the available output of firms to the agents. Since the FOCs (4) for investment involve the agents' utility functions, they do not coincide with the first-order conditions for maximizing firms' profits. The FOCs can be written in an equivalent form which is useful for our analysis by exploiting the separability of agents' preferences.

For fixed $\bar{x}_0 = (\bar{x}_0^i)_{i\in I}$, the function $\Phi_{\bar{x}_0} : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\Phi_{\bar{x}_0}(\eta) = \max\left\{ \sum_{i\in I} \frac{u_1^i(\xi_i)}{u_0^{i\prime}(\bar{x}_0^i)} \,\middle|\, \xi_i \geq 0,\ i \in I,\ \sum_{i\in I} \xi_i = \eta \right\} \tag{5}$$

is called the *sup-convolution* of the $I$ functions $\left(\dfrac{u_1^i}{u_0^{i\prime}(\bar{x}_0^i)}\right)_{i\in I}$. $\Phi_{\bar{x}_0}(\eta)$ is the maximum social welfare that can be attained by distributing $\eta$ units of good to $I$ agents with utility functions $(u_1^i)_{i\in I}$, when the weight of agent $i$ in the social welfare function is $1/u_0^{i\prime}(\bar{x}_0^i)$. Using the first-order conditions and the properties of $u_1^i$, it is easy to see that $(\xi_i^*)_{i\in I} \gg 0$ is solution of (5) if and only if $\sum_{i\in I} \xi_i^* = \eta$ and there exists a scalar $\pi^* > 0$ such that

$$\frac{u_1^{i\prime}(\xi_i^*)}{u_0^{i\prime}(\bar{x}_0^i)} = \pi^*, \qquad i \in I$$

Using the envelope theorem it is then easy to see that $\Phi'_{\bar{x}_0}(\eta) = \pi^*$ (see e.g Magill-Quinzii (1996) for properties of the sup-convolution function).

The FOCs (3) are equivalent to the existence of a vector $\bar{\pi} = (\bar{\pi}_s)_{s\in S} \gg 0$, with $\bar{\pi}_s = \frac{\bar{\lambda}_s}{p_s(\bar{a})\bar{\lambda}_0}$, such that

$$\frac{u_1^{i\prime}(\bar{x}_s^i)}{u_0^{i\prime}(\bar{x}_0^i)} = \bar{\pi}_s, \quad \forall s \in S, \quad \forall i \in I \tag{6}$$

Such a common vector of marginal rates of substitution between date 0 consumption and consumption in outcome $s$ at date 1 is called a vector of *stochastic discount factors*. Given the properties of the function $\Phi_{\bar{x}_0}$ just described, the FOCs (3) imply that, for all $s \in S$, $\Phi_{\bar{x}_0}(Y_s) = \sum_{i\in I} \frac{u_1^i(\bar{x}_s^i)}{u_0^{i\prime}(\bar{x}_0^i)}$, so that the FOCS (3) and (4) are equivalent to the existence of a vector $\bar{\pi}$ of stochastic discount factors satisfying (6) and

$$\sum_{s\in S} \frac{\partial p_s}{\partial a_k}(\bar{a})\Phi_{\bar{x}_0}(Y_s) - 1 = 0, \qquad k \in K \tag{7}$$

To find a concept of equilibrium which leads to Pareto optimal allocations, note that the first-order conditions (7) will hold if each firm $k$ chooses $a_k$ to maximize the objective function $V^k(a_k, \bar{a}^{-k})$ defined by

$$V^k(a_k, \bar{a}^{-k}) = \sum_{s\in S} p_s(a_k, \bar{a}^{-k})\Phi_{\bar{x}_0}(Y_s) - a_k \tag{8}$$

$V^k(a_k, \bar{a}^{-k})$ is the contribution of firm $k$ to the (discounted) expected social utility when its investment is $a_k$ and the investments of other firms are $\bar{a}^{-k}$. The utility function in firm $k$'s objective depends on the preferences of all agents[7] through (5) and depends on aggregate output rather than just the output of firm $k$: this is because the social value of firm $k$'s output depends of the output of the other firms to which it is added. Let $Y_{s^{-k}}^{-k}$ denote the output produced by firms other than $k$ in outcome $s$, i.e. $Y_{s^{-k}}^{-k} = \sum_{k' \neq k} y_{s_{k'}}^{k'}$. If the firms' outputs are independent random variables so that $p_s(a) = \prod_{k \in K} p_{s_k}^k(a_k)$, then $V^k$ can be written as

$$V^k(a_k, \bar{a}^{-k}) = \sum_{s_k \in S_k} p_{s_k}(a_k) \sum_{s^{-k} \in S^{-k}} p_{s^{-k}}(\bar{a}^{-k}) \Phi_{\bar{x}_0}(y_{s_k}^k + Y_{s^{-k}}^{-k}) - a_k$$

or

$$V^k(a_k, \bar{a}^{-k}) = \sum_{s_k \in S_k} p_{s_k}(a_k) \Psi^k(y_{s_k}^k; \bar{a}^{-k}) - a_k$$

where the utility function $\Psi^k$ for the output of firm $k$ is the average value of the social utility obtained by adding $y^k$ to the output of other firms—an average which depends on the probability of the total output of the other firms, and thus on these firms' investments. If the outcomes are correlated rather than independent, $\Psi^k$ will also depend on $a_k$ since it will be a conditional expectation rather than a simple expectation of $\Phi_{\bar{x}_0}(y^k + Y^{-k})$.

As usual it is helpful to decompose the equilibrium of a production economy into two parts—an equilibrium in the allocation of *consumption* among agents which takes place through markets, and an equilibrium in the choice of *investment* by the firms. Consider first an equilibrium in the allocation of consumption: since the investment of firms is taken as given, we may call this a *consumption equilibrium with fixed investment*. Let $P = (P_0, (P_s)_{s \in S})$ denote the vector of prices at date 0 for delivery of income at date 0 and in the different outcomes at date 1: thus $P_s$ (resp $P_0$) is the price at date 0 of a promise to deliver one unit of good (income) at date 1 in outcome $s$ (resp at date 0). It is natural to normalize the prices so that $P_0 = 1$. We let $P_{\mathbf{1}} = (P_s)_{s \in S})$ denote the vector of present-value prices for income at date 1. Thus $P = (1, P_{\mathbf{1}})$.

**Definition 1.** $(\bar{x}, \bar{P}) \in \mathbb{R}_+^{I(S+1)} \times \mathbb{R}_+^{S+1}$ *is a consumption equilibrium with fixed investment* $\bar{a} \in \mathbb{R}_+^K$, *for the economy* $\mathcal{E}(U, w_0, \delta, y, p)$, *if*

(i) *each agent* $i \in I$ *chooses consumption* $\bar{x}^i$ *which maximizes* $u_0^i(x_0^i) + \sum_{s \in S} p_s(\bar{a}) u_1^i(x_s^i)$ *subject to the budget constraint*

$$\bar{P} x^i \leq w_0^i + \sum_{k \in K} \delta_k^i (\bar{P}_{\mathbf{1}} y^k - \bar{a}_k) \tag{9}$$

[7] This is the main difference between the expected utility criterion which emerges in our approach from the first-order condition for optimality, and the objective postulated in Radner (1972). Radner assumed that each firm maximizes the expected utility of its profit, but did not link the exogenously given utility function of the firm to the preferences of the consumers/shareholders.

(ii) *markets clear:* $\sum_{i \in I} \bar{x}_0^i + \sum_{k \in K} \bar{a}_k = \sum_{i \in I} w_0^i, \qquad \sum_{i \in I} \bar{x}_s^i = \sum_{k \in K} y_{s_k}^k, \quad \forall s \in S$

A consumption equilibrium with fixed investment is a standard competitive equilibrium of an exchange economy in which agents have preferences $U^i(x^i, \bar{a})$ and initial resources $(w_0^i - \sum_{k \in K} \delta_k^i \bar{a}_k, \sum_{k \in K} \delta_k^i y^k)$. It can also be interpreted as an equilibrium in which agents trade Arrow securities based on the firms' outcomes or, equivalently, as the reduced form of an equilibrium in which agents have initial equity in the firms and trade securities whose payoffs are based on the profits of the firms—equity contracts, bonds, options, indices on options— which are sufficiently rich to span the outcome space $S$. Thus implicit in Definition 1 is the assumption that the securities in the extensive-form equilibrium satisfy the full spanning condition with respect to outcomes.

Note that in Definition 1, from the point of view of investors the firms' outcomes play exactly the same role as the states of nature in the standard GE model. Since investors take the firms' investment decisions $\bar{a}$ as independent of their trades, the probabilities $(p_s(\bar{a}))$ are fixed and the exchange part of the model is an Arrow-Debreu economy in which uncertainty is modeled by the "states" $s \in S$. Thus while the distinction between states of nature and outcomes is crucial for a production economy in which firms' actions influence outcomes, it is irrelevant for an exchange economy.[8]

We now extend this concept of equilibrium to include the choice of investment by firms. Since firms maximize an expected utility criterion and since this concept leads under appropriate assumptions to first-best optimality, we call it a "strong firm-expected-utility" (SFEU) equilibrium.

**Definition 2.** $(\bar{a}, \bar{x}, \bar{P}) \in \mathbb{R}_+^K \times \mathbb{R}_+^{I(S+1)} \times \mathbb{R}_+^{S+1}$ *is a strong firm-expected-utility equilibrium (SFEU equilibrium) for the production economy* $\mathcal{E}(U, w_0, \delta, y, p)$ *if*

(i) $(\bar{x}, \bar{P})$ *is a consumption equilibrium with fixed investment* $\bar{a}$

(ii) *for each firm* $k \in K$ *the investment* $\bar{a}_k$ *maximizes the expected social utility of its investment*

$$V^k(a_k, \bar{a}^{-k}) = \sum_{s \in S} p_s(a_k, \bar{a}^{-k}) \Phi_{\bar{x}_0}(Y_s) - a_k$$

*where* $\Phi_{\bar{x}_0}$ *is the social utility function defined by (5).*

The expected utility criterion $V^k$ seems rather far removed from the standard criterion for a firm: let us show however that when firms are infinitesimal, it essentially coincides with the standard market-value (present-value-of-profit) criterion.

[8] This probably explains why finance, which takes the payoffs of securities as given when studying asset pricing and portfolio theory, is based on the state-of-nature model.

**Marginal firms: convergence of $V^k$ to market value.** Consider the case where firm $k$'s output is "small" in a sense made precise below and where its investment does not affect the probability of the other firms' outcomes. We formalize this latter no-externality condition in the following assumption:

**Assumption NE (no externality):** *For all $a \in \mathbb{R}_+^K$ and $s \in S$, the probability that the firms different from $k$ have a realization $s^{-k}$ does not depend on $a_k$*

$$\sum_{s_k \in S_k} p_{(s_k, s^{-k})}(a_k, a^{-k}) = p_{s^{-k}}(a^{-k})$$

Firms' outcomes can be stochastically dependent because they are subject to common shocks even though the investment of any firm has no direct effect on the probability of other firms' outcomes. Suppose for example that there is a vector $\gamma$ of unobservable common shocks, with distribution function $H$, which affects the probabilities of the firms' outcomes and that, conditional on $\gamma$, the firms' outcomes are independent. Then

$$p_s(a) = \int p_{s_1}^1(a_1, \gamma) \dots p_{s_K}^K(a_K, \gamma) dH(\gamma)$$

and Assumption NE is satisfied even though the firms' outcomes are stochastically dependent.

Since aggregate output is the sum of firm $k$'s output and the output of all other firms, $Y_s = y_{s_k}^k + Y_{s^{-k}}^{-k}$, the objective function $V^k$ can be expressed using the Taylor formula around $Y^{-k}$: there exists $(\theta_s^k)_{k \in K, s \in S}$ with $0 \leq \theta_s^k \leq 1$ such that

$$V^k(a) = \sum_{s^{-k}} \sum_{s_k} p_{(s_k, s^{-k})}(a_k, a^{-k}) \left( \Phi_{\bar{x}_0}(Y_{s^{-k}}^{-k}) + \Phi'_{\bar{x}_0}(Y_{s^{-k}}^{-k}) \, y_{s_k}^k + \frac{1}{2} \Phi''_{\bar{x}_0}(Y_{s^{-k}}^{-k} + \theta_s^k y_{s_k}^k)(y_{s_k}^k)^2 \right) - a_k \tag{10}$$

Let $m$ denote the bound on the the relative risk aversion.

$$\frac{-\xi \; \Phi''_{\bar{x}_0}(\xi)}{\Phi'_{\bar{x}_0}(\xi)} \leq m$$

of the utility function $\Phi_{\bar{x}_0}$, for $\xi$ lying in the range of values taken by $Y^{-k}$ and $Y$, where we assume that these random variables take values bounded away from 0. We say that firm $k$ is marginal if there exists an $\epsilon > 0$ sufficiently small (see below) such that $\frac{y_{s_k}^k}{Y_{s^{-k}}^{-k}} < \epsilon$ for all $s_k \in S_k$, $s^{-k} \in S^{-k}$. For such a firm the quadratic term in (10) satisfies

$$\left| \frac{1}{2} \Phi''_{\bar{x}_0}(Y_{s^{-k}}^{-k} + \theta_s^k y_{s_k}^k)(y_{s_k}^k)^2 \right| \leq \frac{1}{2} m \, \epsilon \, \Phi'_{\bar{x}_0}(Y_{s^{-k}}^{-k}) \, y_{s_k}^k \tag{11}$$

By Assumption NE the first term on the right side of (10) does not depend on $a_k$ and can thus be omitted from the objective of firm $k$. In view of (11) if $\epsilon$ is sufficiently small, the quadratic term in (10) is negligible relative to the linear term which can be written as

$$E_a(\Phi'_{\bar{x}_0}(Y^{-k})\, y^k) = E_a(\Phi'_{\bar{x}_0}(Y^{-k}))E_a(y^k) + cov_a(\Phi'_{\bar{x}_0}(Y^{-k}), y^k)$$

Since $\epsilon$ is small and $Y^{-k}_{s^{-k}} \leq Y^{-k}_{s^{-k}} + y^k_{s_k} \leq Y^{-k}_{s^{-k}}(1+\epsilon)$, $E_a(\Phi'_{\bar{x}_0}(Y^{-k}))$ is close to $E_a(\Phi'_{\bar{x}_0}(Y)) = \frac{1}{1+\bar{r}}$, where $\bar{r}$ is the interest rate implied by the price vector $\bar{P}$. Thus for a marginal firm the criterion $V^k$ can be replaced by the criterion

$$\widehat{V}^k(a) = E_a\left(\Phi'_{\bar{x}_0}(Y^{-k})\, y^k\right) - a_k = \frac{E_a(y^k)}{1+\bar{r}} + \mathrm{cov}_a\left(\Phi'_{\bar{x}_0}(Y), y^k\right) - a_k \tag{12}$$

Removing the quadratic term in (10) implies that firm $k$ does not worry about its own risk in its choice of investment, but only takes into account the covariance of its output with the stochastic discount factor. The criterion $\widehat{V}^k$ is similar to the criterion which leads to optimality in the state-of-nature approach with complete markets: for $\widehat{V}^k(a) = \sum_{s\in S} p_s(a)\bar{\pi}_s y^k_s - a_k$, so that $\widehat{V}^k$ is just the present value of the profit of firm $k$, or equivalently its market value.

## 4. Non-Marginal Firms and Second-Best Analysis

One of the powerful conclusions of the Arrow-Debreu state-space approach is that profit maximization leads to efficiency, regardless of the size of the firms which are considered, provided the firms are price takers and do not seek to manipulate prices. One may discuss whether the price-taking behavior is realistic for large firms but, in the setting of capital markets, taking security prices as given is widely regarded as a good approximation, even for large corporations. Under these conditions, when the Arrow-Debreu state-space approach is applied to capital markets, all firms, both marginal and non-marginal, should seek to maximize market value. An important corollary of this conclusion is that a firm needs no further information about the preferences and technology of other consumers and firms than that contained in the prices. If, with current and anticipated prices, profit cannot be increased, then the firm's investment is optimal both for its shareholders and for the economy as a whole.

In the probability model the criterion $V^k$ for a firm, which comes from the normative analysis, requires knowing the social welfare function $\Phi_{\bar{x}_0}$ and this in turn requires knowing the utility functions of the consumers. This is a demanding requirement, since revelation of preferences is problematic both because of the amount of information that needs to be transmitted and because of the distortions typically created by incentives. Do prices in the probability model loose all their usefulness for conveying the information about the preferences of consumers? Intuitively this should not be the case since what firms need to know are utilities, or more precisely as we shall see, differences in utilities, and prices signal marginal utilities.

To study the the information that can be conveyed by prices in the probability model, we explicitly introduce the assumption that *firms do not know agents' utility functions*, but seek to maximize a criterion which is their best estimate of the criterion $V^k$. To show how such a criterion can be found, we first introduce the concept of a second-best optimum which explicitly takes into account the informational constraint that agents' utility functions are not known. In the next section we show how to derive the approximate criterion by analyzing the FOCs for a constrained efficient allocation.

To describe the 'best' outcomes that can be achieved when firms do not know the utility functions of consumers, we need to modify the usual concept of Pareto optimality to take this constraint into account. While firms do not have access to direct information on the utility functions $(U^i)_{i \in I}$, they do know something about consumers' preferences since they can observe the prices associated with a consumption equilibrium $(x, P)$, and this gives information on the common marginal rates of substitution of the consumers at the equilibrium consumption $x$. We are thus led to consider allocations $(a, x, P)$ in which the consumption component $x$ can be achieved, for some characteristics of the consumers, by trading on markets at prices $P$ when investment is fixed at $a$. We call such allocations $(a, x, P)$ "constrained feasible" because they incorporate the constraint that the consumption component $x$ is achieved through trading on markets at prices $P$.[9] When the firms' investment $a$ is fixed, the utility function $U^i(x^i, a)$ of an agent satisfying EU is characterized by the pair $(u_0^i, u_1^i)$: we write $U^i = (u_0^i, u_1^i)$ and let $U = (U^i)_{i \in I}$ denote the profile of utility functions of the $I$ consumers. This leads to the following definition.

**Definition 3.** *$(a, x, P)$ is a constrained feasible allocation if for some profile of utility functions $U$ satisfying EU, and of endowments $(w_0, \delta)$ satisfying IN, $(x, P)$ is a consumption equilibrium with fixed investment $a$.*

In the consumption equilibrium $(x, P)$ with fixed investment $a$, the first-order conditions of the maximization problem of the consumers are $\dfrac{p_s(a) u^{i\prime}(x_s^i)}{u^{i\prime}(x_0^i)} = P_s$, $s \in S$, so that the stochastic

[9] Introducing prices in the definition of a constrained feasible allocation is also used in the state-of-nature general equilibrium model with incomplete markets (GEI) to study the best that can be achieved under the constraint that financial markets are incomplete (see Geanakoplos-Polemarchakis (1986), Geanakoplos-Magill-Quinzii-Drèze (1990) and the ensuing literature).

discount factor $\pi$ associated with the equilibrium is given by

$$\pi_s = \frac{P_s}{p_s(a)}, \qquad s \in S \tag{13}$$

The following monotonicity properties of a consumption equilibrium play an important role in the second-best analysis. Let $x^i_{\mathbf{1}} = (x^i_s)_{s \in S}$ denote the date 1 part of the agent's consumption stream $x^i$.

**Proposition 1: (Monotonicity)** *Let $(x, P)$ be a consumption equilibrium with fixed investment $a$ and let $\pi$ be the associated stochastic discount factor. Then*

(i) *each agent's date 1 consumption vector $x^i_{\mathbf{1}}$ is comonotone with date 1 aggregate output $Y$, i.e. $Y_{s'} \geq Y_s$ implies $x^i_{s'} \geq x^i_s$, and $Y_{s'} > Y_s$ implies $x^i_{s'} > x^i_s$, for all $s, s' \in S$*

(ii) *the stochastic discount factor $\pi$ is antimonotone with date 1 aggregate output $Y$, i.e. $Y_{s'} \geq Y_s$ implies $\pi_{s'} \leq \pi_s$, and $Y_{s'} > Y_s$ implies $\pi_{s'} < \pi_s$, for all $s, s' \in S$.*

These are well-known properties of Pareto optimal allocations in economies with separable preferences (see e.g. Magill-Quinzii (1996)): when risk markets are complete, in particular when there are no uninsurable idiosyncratic risks, all agents consume more when aggregate output is high than when aggregate output is low. However the variability of an agent's consumption depends on his/her risk-aversion, the consumption of a risk-tolerant agent varying more than that of a more risk-averse agent. This monotonicity property implies that, for given production of other firms, if firm $k$ produces more, all agents consume more.

Given the Second Theorem of Welfare Economics, the statement that $(a, x, P)$ is a constrained feasible allocation can be replaced by the statement that $(a, x)$ is a feasible allocation and that, for some utility functions, the agents' stochastic discount factors at $x$ are equal to the stochastic discount factor induced by $P$ via (13). We may thus equivalently write a constrained feasible allocation $(a, x, P)$ as the triple $(a, x, \pi)$. For $x \in \mathbb{R}^{(S+1)I}_{++}$ and $\pi \in \mathbb{R}^S_{++}$, let $\mathcal{U}(x, \pi)$ denote the set of utility profiles $U = (U^i)_{i \in I}$ such that for all $i \in I$, $U^i$ satisfies Assumption EU and agent $i$'s stochastic discount factor at $x^i$ is equal to $\pi$,

$$\mathcal{U}(x, \pi) = \left\{ (U^i)_{i \in I} = (u^i_0, u^i_1)_{i \in I} \;\middle|\; \begin{array}{l} U^i \text{ satisfies } EU \\ \dfrac{u^{i\prime}_1(x^i_s)}{u^{i\prime}_0(x^i_0)} = \pi_s, \ \forall\, s \in S \end{array} \right\} \tag{14}$$

**Proposition 2.** $(a, x, \pi)$ *is a constrained feasible allocation if and only if*

(i) *it is feasible with investment* $a$, *i.e*, $\sum_{i\in I} x_0^i + \sum_{k\in K} a_k = \sum_{i\in I} w_0^i, \quad \sum_{i\in I} x_s^i = Y_s, \quad s \in S$

(ii) $\mathcal{U}(x,\pi) \neq \emptyset$.

PROOF: The result follows from the First and Second Welfare Theorems. □

In the standard definition of an efficient allocation, a fictitious planner is assumed to examine the current allocation to consider if there is another feasible allocation which is preferred by all consumers: thus to decide whether or not the current allocation is efficient, the planner must *know the utility functions* of all agents. If firms do not know the preferences of agents—or more precisely, do not know more about consumers' preferences than that they are consistent with the observed stochastic discount factor $\pi$—then to obtain a consistent definition of "constrained efficiency" the planner should be restricted to the same limited information regarding consumers' utility functions. Since the planner knows less about the utility functions of consumers than in the standard setting, the concept of an inefficient allocation needs to be weakened: an allocation $(a, x)$ with observed stochastic discount factor $\pi$ is said to be *inefficient* if there exists another feasible allocation $(\tilde{a}, \tilde{x})$ which dominates the allocation $(a, x)$ *for all conceivable utility functions consistent with the observed* $\pi$, i.e. for all $U \in \mathcal{U}(x,\pi)$. More precisely

**Definition 4.** *A constrained feasible allocation* $(a, x, \pi)$ *is inefficient if there exists an allocation* $(\tilde{a}, \tilde{x})$ *with* $\sum_{i\in I} \tilde{x}_0^i + \sum_{k\in K} \tilde{a}_k = \sum_{i\in I} w_0^i$, $\sum_{i\in I} \tilde{x}_s^i = Y_s$, $s \in S$, *such that, for every profile of utility functions* $(U^i)_{i\in I} = (u_0^i, u_1^i)_{i\in I} \in \mathcal{U}(x,\pi)$

$$u_0^i(\tilde{x}_0^i) + \sum_{i\in I} p_s(\tilde{a}) u_1^i(\tilde{x}_s^i) > u_0^i(x_0^i) + \sum_{i\in I} p_s(a) u_1^i(x_s^i), \quad \forall\, i \in I$$

*A constrained feasible allocation which is not inefficient is said to be constrained Pareto optimal.*

This definition incorporates the constraint that the planner only has limited information regarding the preferences of the agents when he seeks to change the current allocation to one that improves the welfare of all agents: he must be sure to improve the allocation for all potential utility functions consistent with the observed vector of prices $P$ (or equivalently stochastic discount factor $\pi$) at the current consumption allocation $x$. Since for fixed $a$, by the First Welfare Theorem, the distribution of output among agents is efficient, the only possible source of inefficiency is an inappropriate choice of investment at date 0. Note that only a standard feasibility constraint is imposed on the dominating allocation: the planner does not need to respect market prices or agents' budget constraints in reallocating goods. Also, since agents have strictly monotone preferences, w.l.o.g. a social improvement is defined as a strict improvement for every agent in the economy.

We will see that some similarity in the use of information by firms is needed to achieve a constrained Pareto optimal allocation: otherwise firms' investment choices are only efficient in the following weaker sense.

**Definition 5.** *A constrained feasible allocation $(a, x, \pi)$ is firm $k$-inefficient if there exists an allocation $(\tilde{a}, \tilde{x})$ with $\tilde{a}_{k'} = a_{k'}$ if $k' \neq k$, $\sum_{i \in I} \tilde{x}_0^i + \sum_{k \in K} \tilde{a}_k = \sum_{i \in I} w_0^i$, $\sum_{i \in I} \tilde{x}_s^i = Y_s, s \in S$, such that, for every profile of utility functions $(U^i = (u_0^i, u_1^i), i \in I) \in \mathcal{U}(x, \pi)$*

$$u_0^i(\tilde{x}_0^i) + \sum_{i \in I} p_s(\tilde{a}) u_1^i(\tilde{x}_s^i) > u_0^i(x_0^i) + \sum_{i \in I} p_s(a) u_1^i(x_s^i), \quad \forall\, i \in I$$

*A constrained feasible allocation which is not firm $k$-inefficient is said the be firm $k$-efficient.*

An allocation is $k$-inefficient if it is possible to improve on it by changing the investment of firm $k$ and the allocation to the consumers, leaving the investments of all other firms unchanged.

## 5. Estimated Social Utility and FEU Equilibrium

Constrained efficient allocations cannot be found by maximizing a social welfare function since this would require knowing the utility functions of the agents. We thus proceed directly by studying whether there are marginal changes $(da, dx)$ from a constrained feasible allocation which can increase the utilities of all agents, for all possible profiles of their utility functions consistent with the observed prices, i.e. for all $U$ in $\mathcal{U}(x, \pi)$.

The analysis will make repeated use of the following relation which is the discrete equivalent of integration by parts. Let $X$ be a discrete random variable taking values $(X_1 < X_2 < \ldots < X_S)$ with probabilities $(p_s)_{s \in S}$, and let $F$ and $G = 1 - F$ denote the associated distribution function and upper cumulative distribution function

$$F(x) = \sum_{\{s \in S | X_s \leq x\}} p_s, \qquad G(x) = \sum_{\{s \in S | X_s > x\}} p_s$$

It is easy to verify that if $h : \mathbb{R} \to \mathbb{R}$ is a real valued function

$$E(h(X)) = \textstyle\sum_{s \in S} p_s h(X_s) = h(X_1) + \sum_{s=1}^{S-1} G(X_s) \left( h(X_{s+1}) - h(X_s) \right) \tag{IP}$$

which we will refer to as the *integration by parts* relation.

The analysis also makes use of the monotonicity properties of a constrained feasible allocation with respect to aggregate output described in Proposition 1. To exploit these monotonicity properties it is useful to order the outcome space $S = \prod_{k=1}^{K} S_k$ by increasing values of the date 1 aggregate output $Y = \sum_{k\in K} y^k$. More precisely the random variable $Y$ induces a partition of the outcome space $S$ into equivalence classes on which the aggregate output is constant

$$s \sim s' \quad \text{if} \quad Y_s = \sum_{k\in K} y^k_{s_k} = \sum_{k\in K} y^k_{s'_k} = Y_{s'}$$

Let $\Sigma$ denote the set of distinct values of $Y$: for any $\sigma \in \Sigma$, $s, s' \in S$ lie in the same equivalence class ($s, s' \in \sigma$) if $Y_s = Y_{s'} = Y_\sigma$. Without loss of generality we can order the elements of $\Sigma$ in increasing order for $Y$: $\sigma > \sigma' \Longrightarrow Y_\sigma > Y_{\sigma'}$. We let $\underline{Y}$ and $\overline{Y}$ denote the smallest and the largest values of $Y$ respectively: $\underline{Y} = \sum_{k\in K} y^k_1$, $\overline{Y} = \sum_{k\in K} y^k_{S_k}$.

The probability that aggregate output is equal to $Y_\sigma$ when the firms' vector of investment is $a$ is

$$p_\sigma(a) = \sum_{\{s\in S \,|\, Y_s = Y_\sigma\}} p_s(a), \quad \sigma \in \Sigma$$

Let $F(\eta, a)$ and $G(\eta, a)$ denote the distribution function and upper distribution functions of $Y$ given $a$ defined by

$$F(\eta, a) = \sum_{\{\sigma\in\Sigma | Y_\sigma \le \eta\}} p_\sigma(a), \qquad G(\eta, a) = 1 - F(\eta, a)$$

Consider a constrained feasible allocation $(\bar{a}, \bar{x}, \bar{\pi})$. Since $(\bar{x}, \bar{P})$, with $\bar{P}_s = p_s(\bar{a})\bar{\pi}_s$, is a consumption equilibrium with fixed investment, by Proposition 1, $\bar{x}$ and $\bar{\pi}$ only depend on $\sigma$: $Y_s = Y_{s'} = Y_\sigma \Longrightarrow \bar{x}_s = \bar{x}_{s'} \stackrel{\text{def}}{=} \bar{x}_\sigma$, $\bar{\pi}_s = \bar{\pi}_{s'} \stackrel{\text{def}}{=} \bar{\pi}_\sigma$, and $\bar{x}$ increases with $\sigma$ while $\bar{\pi}$ decreases with $\sigma$. The consumption equilibrium $(\bar{x}, \bar{P})$ has associated with it a social welfare function $W : \mathbb{R}^{I(S+1)}_+ \times \mathbb{R}^K_+ \to \mathbb{R}$

$$W(x, a) = \sum_{i\in I} W^i(x^i, a), \qquad W^i(x^i, a) = \frac{1}{u^{i\prime}(\bar{x}^i_0)} U^i(x^i, a),\ i \in I$$

which is maximized at $\bar{x}$ when $a = \bar{a}$. To see whether it is possible to make a marginal improvement from $(\bar{a}, \bar{x})$ by changing firm $k$'s investment, consider a marginal change $da_k$ followed by a reallocation $(dx^i)_{i\in I}$ of the agents' consumption streams: such a change is feasible if $\sum_{i\in I} dx^i_0 + da_k \le 0$, $\sum_{i\in I} dx^i_s \le 0$. If there is a feasible change $(da_k, (dx^i)_{i\in I})$ such that $dW^i > 0$, for all $i$ and every profile $(U^i)_{i\in I} \in \mathcal{U}(\bar{x}, \bar{\pi})$, then $(\bar{a}, \bar{x}, \bar{\pi})$ is constrained inefficient.

There is no loss of generality in assuming that the date 1 change $dx^i_{\mathbf{1}} = 0$ for all agents since the date 1 aggregate resources do not change and these resources are shared efficiently at $\bar{x}$: there is no possibility of increasing the utility of all agents by redistributing date 1 output. Then

$$dW^i = dx^i_0 + \frac{1}{u^{i\prime}_0(\bar{x}^i_0)} \sum_{\sigma\in\Sigma} \frac{\partial}{\partial a_k} p_\sigma(\bar{a}) u^i_1(\bar{x}^i_\sigma) da_k$$

which, in view of the (IP) relation , can be written as

$$dW^i = dx_0^i + \sum_{\sigma=1}^{\Sigma-1} G_{a_k}(Y_\sigma, \bar{a}) \frac{u_1^i(\bar{x}_{\sigma+1}^i) - u_1^i(\bar{x}_\sigma^i)}{u_0^{i\prime}(\bar{x}_0^i)} da_k$$

where $G_{a_k}$ denotes the partial derivative of the upper cumulative function $G$ with respect to $a_k$. Given the monotonicity properties of $\bar{x}$, even though the planner does not know $U^i$ he can use the stochastic discount factor to obtain bounds on the difference $\frac{u_1^i(\bar{x}_{\sigma+1}^i)-u_1^i(\bar{x}_\sigma^i)}{u_0^{i\prime}(\bar{x}_0^i)}$. Since $u_1^i$ is concave and $u_1^i(\bar{x}_{\sigma+1}^i) - u_1^i(\bar{x}_\sigma^i) = \int_{\bar{x}_\sigma^i}^{\bar{x}_{\sigma+1}^i} u_1^{i\prime}(t)dt$, it follows that

$$\bar{\pi}_{\sigma+1}(\bar{x}_{\sigma+1}^i - \bar{x}_\sigma^i) < \frac{u_1^i(\bar{x}_{\sigma+1}^i) - u_1^i(\bar{x}_\sigma^i)}{u_0^{i\prime}(\bar{x}_0^i)} < \bar{\pi}_\sigma(\bar{x}_{\sigma+1}^i - \bar{x}_\sigma^i) \tag{15}$$

Let $\overline{W_{a_k}^i}$ and $\underline{W_{a_k}^i}$ respectively denote the supremum and the infimum of $W_{a_k}^i$ for all admissible utility functions $U^i$ satisfying Assumption EU and such that the stochastic discount factor at $\bar{x}^i$ is $\bar{\pi}$. To find $\overline{W_{a_k}^i}$ and $\underline{W_{a_k}^i}$, we use (15) and define the two subsets of $\Sigma$

$$\Sigma_+ = \{\sigma \in \Sigma \,|\, G_{a_k}(Y_\sigma, \bar{a}) > 0\}, \qquad \Sigma_- = \{\sigma \in \Sigma \,|\, G_{a_k}(Y_\sigma, \bar{a}) < 0\} \tag{16}$$

where for simplicity we omit the dependence of the sets on $k$, since firm $k$ is fixed for this local analysis. $\Sigma_+$ is the set of outcomes $\sigma$ such that a marginal increase $da_k$ in the investment of firm $k$ from $\bar{a}_k$ increases the probability that the production is greater than or equal to $Y_\sigma$, while $\Sigma_-$ is the set of outcomes for which the inequality is reversed. If, for example, increasing the investment of the firm leads to a first-order stochastic dominant shift in the distribution of its output, then $\Sigma_+ = \{1, \ldots, \Sigma-1\}$, and $\Sigma_- = \emptyset$. If increasing investment $a_k$ only leads to a second-order stochastic dominant shift in the distribution of its output then both $\Sigma_+$ and $\Sigma_-$ may be non empty. With this notation,

$$\overline{W_{a_k}^i} = \sum_{\Sigma_+} G_{a_k}(Y_\sigma, \bar{a})\bar{\pi}_\sigma(\bar{x}_{\sigma+1}^i - \bar{x}_\sigma^i) + \sum_{\Sigma_-} G_{a_k}(Y_\sigma, \bar{a})\bar{\pi}_{\sigma+1}(\bar{x}_{\sigma+1}^i - \bar{x}_\sigma^i)$$

$$\underline{W_{a_k}^i} = \sum_{\Sigma_+} G_{a_k}(Y_\sigma, \bar{a})\bar{\pi}_{\sigma+1}(\bar{x}_{\sigma+1}^i - \bar{x}_\sigma^i) + \sum_{\Sigma_-} G_{a_k}(Y_\sigma, \bar{a})\bar{\pi}_\sigma(\bar{x}_{\sigma+1}^i - \bar{x}_\sigma^i)$$

The bounds on the marginal benefit of agent $i$ from a marginal change $(da_k, dx)$ are then

$$dx_0^i + \underline{W_{a_k}^i} da_k \ < dW^i < \ dx_0^i + \overline{W_{a_k}^i} da_k \quad \text{if } da_k > 0 \tag{17}$$

$$dx_0^i + \overline{W_{a_k}^i} da_k \ < dW^i < \ dx_0^i + \underline{W_{a_k}^i} da_k \quad \text{if } da_k < 0 \tag{18}$$

Suppose that $\sum_{i\in I}\underline{W^i_{a_k}} \geq 1$. Consider a change with $da_k > 0$, $dx^i_0 = -\underline{W^i_{a_k}}\,da_k$ (so that $dx^i_0 + \underline{W^i_{a_k}}da_k = 0$). Since $0 = \sum_{i\in I} dx^i_0 + \sum_{i\in I}\underline{W^i_{a_k}}da_k \geq \sum_{i\in I} dx^i_0 + da_k$, the change is feasible and, from (17), $dW^i > 0$ for all $i$. In the same way, if $\sum_{i\in I}\overline{W^i_{a_k}} \leq 1$, consider a change with $da_k < 0$, $dx^i_0 = -\overline{W^i_{a_k}}\,da_k$. The change is feasible and from (18), $dW^i > 0$ for all $i \in I$. We have thus proved the following proposition:

**Proposition 3: (FOC for CPO)** *Let $(\bar{a}, \bar{x}, \bar{\pi})$ be a constrained feasible allocation, let $\Sigma_+$ and $\Sigma_-$ denote the subsets of the aggregate outcomes $\Sigma$ defined by (16), and let*

$$\overline{W_{a_k}} = \sum_{\Sigma_+} G_{a_k}(Y_\sigma, \bar{a})\bar{\pi}_\sigma(Y_{\sigma+1} - Y_\sigma) + \sum_{\Sigma_-} G_{a_k}(Y_\sigma, \bar{a})\bar{\pi}_{\sigma+1}(Y_{\sigma+1} - Y_\sigma)$$

$$\underline{W_{a_k}} = \sum_{\Sigma_+} G_{a_k}(Y_\sigma, \bar{a})\bar{\pi}_{\sigma+1}(Y_{\sigma+1} - Y_\sigma) + \sum_{\Sigma_-} G_{a_k}(Y_\sigma, \bar{a})\bar{\pi}_\sigma(Y_{\sigma+1} - Y_\sigma)$$

*If $(\bar{a}, \bar{x}, \bar{\pi})$ is constrained Pareto optimal, then*

$$\underline{W_{a_k}} < 1 < \overline{W_{a_k}} \tag{19}$$

$\underline{W_{a_k}}$ represents the minimal present value of the social gain in expected utility of consumption at date 1 from an additional unit of investment by firm $k$ at date 0. If this minimal present-value gain exceeds its marginal cost, which is 1, then it is worthwhile to increase investment: hence the inequality $\underline{W_a} < 1$ in (19). $\overline{W_a}$ is the maximal present value of the social gain in expected utility of consumption at date 1 from an additional unit of investment by firm $k$ at date 0 or, in absolute value, the maximal present value of social loss in expected utility of consumption associated with one unit decrease in its investment. If this loss is less that the marginal reduction in cost, then it is worthwhile to decrease investment: hence the inequality $1 < \overline{W_a}$ in (19).

As we saw in Section 3 the market-value criterion is close to the FEU criterion (8) when firms are marginal. The inequality (19) is sufficient to give the sign of the bias in investment when the market-value criterion is used by a non-marginal firm in place of the expected social utility criterion (8). Let

$$M(a_k, \bar{a}^{-k}) = \sum_{s\in S} p_s(a_k, \bar{a}^{-k})\bar{\pi}_s y^k_{s_k} - a_k \tag{20}$$

denote the firm's market value viewed as a function of its investment $a_k$. The natural competitive assumption for this model is that the firm takes the investment $\bar{a}^{-k}$ of the other firms as given as well as the stochastic discount factor $\bar{\pi}$. We show that if there are no external effects among firms

(Assumption NE of Section 3) and if each firm's investment is productive in the sense of first-order stochastic dominance, then maximization of market value leads to underinvestment.

**Assumption FOSD$_k$ (first-order stochastic dominance):** *Let* $p_{s_k}(a_k \mid s^{-k}, a^{-k}) = \dfrac{p_{(s_k,s^{-k})}(a_k, a^{-k})}{p_{s^{-k}}(a^{-k})}$ *denote the conditional probability of* $s_k$ *given* $(s^{-k}, a^{-k})$*. For fixed* $(s^{-k}, a^{-k})$*, an increase in* $a_k$ *leads to a first-order stochastic dominant shift in the conditional probability of* $s_k$*, i.e.*

$$\tilde{a}_k > a_k \Longrightarrow \sum_{\{s_k | y^k_{s_k} > \alpha\}} p_{s_k}(\tilde{a}_k \mid s^{-k}, a^{-k}) > \sum_{\{s_k | y^k_{s_k} > \alpha\}} p_{s_k}(a_k \mid s^{-k}, a^{-k}), \quad \forall\, \alpha \in \left[\underline{y^k},\ \overline{y^k}\right)$$

**Proposition 4: (Market Value not CPO).** *Under Assumptions NE and FOSD$_k$, if* $(\bar{a}, \bar{x}, \bar{\pi})$ *is a constrained feasible allocation such that* $\bar{a}_k$ *is positive and maximizes the market value* $M(a_k, \bar{a}^{-k})$*, then* $\underline{W_{a_k}} > 1$.

**Proof:** see Appendix.

As we saw above, $\underline{W_{a_k}} > 1$ implies that a marginal increase in $a_k$, with an appropriate distribution of the additional date 0 cost among the agents ( $dx^i_0 = -\underline{W^i_{a_k}}\, da_k, i \in I$) increases the utility of all agents for every possible profile of utility functions in $\mathcal{U}(\bar{x}, \bar{\pi})$. Thus Propositon 4 asserts that market-value maximization leads to underinvestment.

**Estimated Social Utility.** Can we find a criterion for the firms, which uses only available information, and which when maximized leads to a constrained efficient investment? Such a criterion must incorporate the information on preferences conveyed by the stochastic discount factor $\bar{\pi}$ at a constrained feasible allocation. By Proposition 1, $\bar{\pi}_\sigma$ is decreasing in $\sigma$. Thus there exists a continuous decreasing function $\phi^k : [\underline{Y},\ \overline{Y}] \to \mathbb{R}$ of aggregate output which coincides with the discount factor $\bar{\pi}_\sigma$ for each value $Y_\sigma$ in $[\underline{Y},\ \overline{Y}]$, i.e. $\phi^k(Y_\sigma) = \bar{\pi}_\sigma, \sigma \in \Sigma$. The simplest example of such a function is the linear interpolation

$$\phi^k(\eta) = \bar{\pi}_\sigma + \frac{\eta - Y_\sigma}{Y_{\sigma+1} - Y_\sigma}(\bar{\pi}_{\sigma+1} - \bar{\pi}_\sigma) \quad \text{if } \eta \in\ [Y_\sigma,\ Y_{\sigma+1}], \quad \sigma = 1, \dots, \Sigma - 1 \tag{21}$$

If firm $k$ chooses its investment $\bar{a}_k$ so as to maximize the objective function

$$\widetilde{V}^k(a_k, \bar{a}^{-k}) = \int_{\underline{Y}}^{\overline{Y}} G(\eta, a_k, \bar{a}^{-k}) \phi^k(\eta) d\eta - a_k \tag{22}$$

then condition (19) for a constrained optimal choice of investment will be satisfied. For $\bar{a}_k$ will satisfy the first-order condition

$$1 = \int_{\underline{Y}}^{\overline{Y}} G_{a_k}(\eta, \bar{a}_k, \bar{a}^{-k})\phi^k(\eta)d\eta = \sum_{\sigma=1}^{\Sigma-1} G_{a_k}(Y_\sigma, \bar{a}) \int_{Y_\sigma}^{Y_{\sigma+1}} \phi^k(\eta)d\eta$$

where we have exploited the property that $G$ is constant on each interval $[Y_\sigma,\ Y_{\sigma+1})$. Since $\bar{\pi}_{\sigma+1} < \phi^k(\eta) < \bar{\pi}_\sigma$ whenever $\eta \in (Y_\sigma,\ Y_{\sigma+1})$, it follows that

$$\bar{\pi}_{\sigma+1}(Y_{\sigma+1} - Y_\sigma) < \int_{Y_\sigma}^{Y_{\sigma+1}} \phi^k(\eta)d\eta < \bar{\pi}_\sigma(Y_{\sigma+1} - Y_\sigma), \qquad \sigma = 1, \ldots, \Sigma - 1$$

which implies that inequality (19) is satisfied. To express the objective function $\widetilde{V}^k$ in a more recognizable form, let us integrate $\phi^k$ to obtain the function

$$\Phi^k : [\underline{Y},\ \overline{Y}] \to \mathbb{R}, \qquad \Phi^k(\eta) = \int_{\underline{Y}}^{\eta} \phi^k(t)dt \tag{23}$$

which is normalized so that $\Phi^k(\underline{Y}) = 0$. $\Phi^k$ is a differentiable, concave, increasing function of aggregate output. Using the (IP) relation, the objective function (22) can be written as

$$\widetilde{V}^k(a_k, \bar{a}^{-k}) = \sum_{\sigma=1}^{\Sigma-1} G(Y_\sigma, \bar{a}) \Big(\Phi^k(Y_{\sigma+1}) - \Phi^k(Y_\sigma)\Big) - a_k = \sum_{\sigma\in\Sigma} p_\sigma(a_k, \bar{a}^{-k})\Phi^k(Y_\sigma) - a_k \tag{24}$$

Thus if $\bar{a}_k$ has been chosen to maximize (22), then the firm has chosen its investment to maximize its contribution to the expected discounted utility $\Phi^k$ of date 1 aggregate output, net of the cost of investment. Using the relation $p_\sigma(a) = \sum_{s\in\sigma} p_s(a)$ and $Y_s = Y_\sigma$ for $s \in \sigma$, the function (24) can be expressed equivalently as an expected utility on the outcome space $S$

$$\widetilde{V}^k(a_k, \bar{a}^{-k}) = \sum_{s\in S} p_s(a_k, \bar{a}^{-k})\Phi^k(Y_s) - a_k \tag{25}$$

$\widetilde{V}^k$ is firm $k$'s estimate of the expected social utility of its investment in the following sense. When agents' utility functions are not known, the sup-convolution $\Phi_{\bar{x}_0}$ of the utility functions $\left(\frac{u_1^i}{u_0^{i\prime}(\bar{x}_0^i)}\right)_{i\in I}$ is not known, but its derivative at the output levels $Y_\sigma, \sigma \in \Sigma$, is equal to the known discount factor $(\bar{\pi}_\sigma)_{\sigma\in\Sigma}$. Firm $k$ chooses a function $\phi^k$ which coincides with these known values $\pi_\sigma$ at the points $Y_\sigma$, $\phi^k(Y_\sigma) = \bar{\pi}_\sigma = \Phi'_{\bar{x}_0}(Y_\sigma)$, "filling in" for the unknown values of the derivative of the social utility on the intervals $(Y_\sigma,\ Y_{\sigma+1})$. This estimate $\phi^k$ of the derivative is then integrated to obtain an estimate of the social utility function $\Phi_{\bar{x}_0}$. There are of course many ways of passing a continuous decreasing function through the points $((Y_\sigma, \bar{\pi}_\sigma), \sigma \in \Sigma)$ and each will lead to a different estimate $\Phi^k$ of $\Phi_{\bar{x}_0}$. The non-uniqueness of $\Phi^k$ implies that there can be many possible "second best" investment levels.

Under Assumption NE, $\widetilde{V}^k$ can be expressed as an expected utility of firm $k$'s output

$$\widetilde{V}^k(a_k, \bar{a}^{-k}) = \sum_{s_k \in S_k} p_{s_k}(a_k)\Psi^k(y^k_{s_k}; a_k, \bar{a}^{-k})$$

the function $\Psi^k$ being defined by

$$\Psi^k(y^k_{s_k}, a_k, \bar{a}^{-k}) = \sum_{s^{-k}} \frac{p(s^{-k}, \bar{a}^{-k}|s_k, a_k)}{p_{s_k}(a_k)} \Phi^k(y^k_{s_k} + Y^{-k}_{s^{-k}})$$

$\Psi^k$ evaluates the average social utility obtained from the firm's output $y^k_{s_k}$, when it is added to the total output $Y^{-k}$ of the other firms.

We are thus led to a concept of equilibrium in which firms are not required have more information than that which can be deduced from the equilibrium prices.

**Definition 6.** *Let $\mathcal{E}(U, w_0, \delta, y, p)$ be an economy with $K$ firms satisfying Assumptions FS, EU, and IN. $(\bar{a}, \bar{x}, \bar{P})$ is an firm-expected-utility (FEU ) equilibrium if*

(i) *$(\bar{x}, \bar{P})$ is a consumption equilibrium with investment $\bar{a}$ and associated discount factor $\bar{\pi}$*

(ii) *each firm $k \in K$ chooses its investment $\bar{a}_k$ to maximize its estimate of the expected social utility of its investment*

$$\widetilde{V}^k(a_k, \bar{a}^{-k}) = \sum_{s \in S} p_s(a_k, \bar{a}^{-k})\Phi^k(Y_s) - a_k$$

*where $\Phi^k : \mathbb{R}_+ \to \mathbb{R}$ is a concave, increasing, differentiable function such that $\Phi^{k\prime}(Y_s) = \bar{\pi}_s$ for all $s \in S$.*

In a strong FEU equilibrium each firm uses the same social utility function $\Phi_{\bar{x}_0}$ which is the sup-convolution function of the known utility functions $\left(\frac{u^i_1(\xi_i)}{u^{i\prime}_0(\bar{x}^i_0)}\right)_{i \in I}$, while in the second-best setting, since firms do not know agents' utilities, each firm derives its estimate $\Phi^k$ of the social utility $\Phi_{\bar{x}_0}$ to form its expected utility criterion $\widetilde{V}^k$. The different estimates $\Phi^k$ of the social utility function, which in essence correspond to the different profiles of utility functions in $\mathcal{U}(\bar{x}, \bar{\pi})$ that agents could have, generate the upper and lower bounds $\overline{W_{a_k}}$ and $\underline{W_{a_k}}$ on the marginal benefit of firm $k$'s investment which appear in the necessary condition (19) for constrained optimality. The finer the partition $\Sigma$ of the interval $[\underline{Y}, \overline{Y}]$, the smaller the possible differences in the estimated social utility functions $\Phi^k$, and the closer an FEU equilibrium will lie to an SFEU equilibrium. If the random variable $Y$ were continuous instead of discrete, and there were a continuum of markets indicating the value of the discount factor $\bar{\pi}(Y)$ for every value of $Y$, then no estimation would be necessary and the function $\Phi_{\bar{x}_0}$ could be constructed from the observed equilibrium prices. To avoid the unrealistic assumption of a continuum of markets we have adopted a discrete model to study the properties of an economy in which investment affects probabilities.

**Will Firms Use the Objective Function $\widetilde{V}^k$?** An FEU equilibrium has two parts to it: agents' choices of consumption $\bar{x}$ and firms' choices of investment $\bar{a}$. The choices in these two components use market prices differently: we can suggestively write $(\bar{x}, \bar{P})$ and $(\bar{a}, (\Phi^k)_{k\in K})$ to indicate that consumption choices are made in a standard way using prices $\bar{P}$, while investment choices use the estimated social utility functions $(\Phi^k)_{k\in K}$ constructed from the prices $\bar{P}$. Can these two distinct uses of prices in consumption and investment be a source of tension? After all, a firm's choice of investment determines not only how probability is spread across date 1 outcomes, but also the profit paid to its shareholders—perhaps initial owners would prefer that the firm focus on market value rather than on the expected social utility of its investment.

Let $(\bar{a}, \bar{x}, \bar{P})$ be an FEU equilibrium with associated stochastic discount factor $\bar{\pi}$. To study the impact on a typical agent of a small change in firm $k$'s investment, let

$$x^i(a) = \operatorname{argmax}\left\{ U^i(x^i, a) \;\middle|\; x_0^i + \sum_{s\in S} p_s(a)\bar{\pi}_s x_s^i = w_0^i + \sum_{k\in K} \delta_k^i M^k(a) \right\}$$

denote the optimal consumption viewed as a function of $a$, where

$$M^k(a) = \bar{P}_{\mathbf{1}} y^k - a_k = \sum_{s\in S} p_s(a)\bar{\pi}_s y^k_{s_k} - a_k$$

is the market value of firm $k$, $k \in K$, and let $U^{i*}(a) = U^i(x^i(a), a)$ denote the agent's optimized utility. If $\bar{\lambda}_i$ denotes the multiplier associated with agent $i$'s budget constraint in the FEU equilibrium, and if we use the competitive assumption that agents (like firms) do not perceive the effect of a change in investment on $\bar{\pi}$, then, by the Envelope Theorem, the derivative of $U^{i*}(a)$ with respect to $a_k$ is

$$U^{i*}_{a_k}(\bar{a}) = \sum_{s\in S} \frac{\partial p_s}{\partial a_k}(\bar{a}) u^i(\bar{x}_s^i) - \bar{\lambda}_i \left( \sum_{s\in S} \frac{\partial p_s}{\partial a_k}(\bar{a})\bar{\pi}_s \bar{x}_s^i - \sum_{k'\in K} \delta^i_{k'} M^{k'}_{a_k}(\bar{a}) \right)$$

where $U^{i*}_{a_k}$ and $M^{k'}_{a_k}$ denote the partial derivative of the utility function $U^i$ and the market value $M^{k'}$ with respect to $a_k$. To simplify the discussion, assume that firms' outcomes are independent (an assumption stronger than NE) and that increasing investment in firm $k$ leads to a first-order stochastic dominant shift in the distribution of its output and thus of aggregate output (FOSD$_k$).

In this case a change in $a_k$ does not affect the market value of other firms $M_{a_k}^{k'} = 0$ for $k' \neq k$. Since in equilibrium $u^{i\prime}(\bar{x}_s^i) = \bar{\lambda}_i \bar{\pi}_s$, the change in welfare can be written as

$$U_{a_k}^{i*}(\bar{a}) = \sum_{s \in S} \frac{\partial p_s}{\partial a_k}(\bar{a}) H^i(\bar{x}_s^i) + \bar{\lambda}_i \delta_k^i M_{a_k}^k(\bar{a})$$

where $H^i(\xi) = u_1^i(\xi) - u^{i\prime}(\xi)\xi$, for $\xi \in \mathbb{R}_+$. Note that $H^i$ is an increasing function since $H^{i\prime}(\xi) = -u^{i\prime\prime}(\xi)\xi > 0$. By Proposition 1, $U_{a_k}^{i*}$ can be written as

$$U_{a_k}^{i*}(\bar{a}) = \sum_{\sigma \in \Sigma} \frac{\partial p_\sigma}{\partial a_k}(\bar{a}) H^i(\bar{x}_\sigma^i) + \bar{\lambda}_i \delta_k^i M_{a_k}^k(\bar{a}) = \sum_{\sigma=1}^{\Sigma-1} G_{a_k}(Y_\sigma, \bar{a}) \left( H^i(\bar{x}_{\sigma+1}^i) - H^i(\bar{x}_\sigma^i) \right) + \bar{\lambda}_i \delta_k^i M_{a_k}^k(\bar{a}) \tag{26}$$

By FOSD$_k$, $G_{a_k}(Y_\sigma, \bar{a})$ is positive so that the first term in the decomposition of $U_{a_k}^{i*}$ is positive. As is shown in the proof of Proposition 5 in next section, there exist utility functions $U^i$ such that $\bar{a}_k$ maximizes $\sum_{i \in I} U^{i*}(a_k, \bar{a}^{-k})/\bar{\lambda}^i$. Since for these utility functions $\sum_{i \in I} U_{a_k}^{i*}(\bar{a})/\bar{\lambda}^i$ is equal to zero, it follows that $M_{a_k}^k(\bar{a}) < 0$.

Thus for each agent the change in welfare from an increase in investment by firm $k$ involves a positive term—the expected utility of consumption at date 1 increases faster than its cost—and, if the agent is an initial owner of the firm, a negative term—increasing investment decreases the market value of the firm, and hence the income that the agent receives as a shareholder. A consumption equilibrium with fixed investment (in Definition 1, or in part (i) of Definition 6) can be thought of as an equilibrium on financial markets in which agents trade securities whose payoffs are based on the observable outputs of the firms—securities such as bonds, equities, options—which satisfy full spanning with respect to the outcomes. It follows from the monotonicity property of the allocations in Proposition 1 that, after trade, every agent will hold a positive share of each firm, so that all the agents can be considered as "new" (i.e. after trade) shareholders of firm $k$ in the sense of Drèze (1974), even if there are not initial shareholders. For the particular profile of utility functions for which $\bar{a}_k$ maximizes $\sum_{i \in I} U^{i*}(a_k, \bar{a}^{-k})/\bar{\lambda}^i$, if $\bar{a}_k$ were announced before the trading phase and if unanimity of the new shareholders is required to change the production plan, then $\bar{a}_k$ would not be changed, since the "winners" from a change $da_k$ would need to spend all their surplus to buy the votes of the agents in favor of the status-quo (i.e. to compensate the loosers from the change). Thus if each agent had no more knowledge than the firm regarding the preferences of other investors, there will not be a coalition of new shareholders who can plan to convene a meeting of all shareholders and be sure to obtain unanimity for a change in the investment plan.[10]

[10] The criterion (25) has the same stability property than the Drèze criterion for the GEI model with incomplete market (Drèze (1974)): since it is derived from constrained optimality, it maximizes the surplus of the agents who

However if the investment decision has to meet with the approval of the initial shareholders and if the initial ownership is not spread among all the agents, the sum $\sum_{\{i\in I|\delta^i>0\}} U^{i*}_{a_k}$ restricted to the initial owners is negative since some of the positive terms $\sum_\sigma G_{a_k}(Y_\sigma,\bar{a})\left(H^i(\bar{x}^i_{\sigma+1})-H^i(\bar{x}^i_\sigma)\right)$ are missing. In this case there may be unanimity among the initial shareholders to decrease investment and increase the market value of the firm. This situation is however artificial since the initial situation is qualitatively different from that in the equilibrium, where the ownership of firms is spread among all agents of the economy. If the two-period model is viewed as a simplified description of a "slice" of an on-going economy, it is more natural to consider dispersed initial ownership than concentrated initial ownership for the type of public corporations to which this study applies.[11]

This being said, the stability property of $\bar{a}_k$ is rather weak. If the investment decision is made by a subgroup of agents for whom the market-value terms in (26) dominate the terms in $H^i$, the firm is more likely to maximize market value than the expected social utility of its profit. Thus an FEU equilibrium is probably better interpreted as a normative benchmark for what firms "should do" rather than a positive description of what they will do in practice. If this is the case, then the analysis of this paper highlights circumstances where subsidization of the investment of some firms may be welfare improving for the economy.

## 6. Normative Properties and Existence of FEU Equilibrium

Without making any further assumptions it can be shown that an FEU equilibrium is firm-$k$ optimal, for any firm $k \in K$: additional assumptions are needed to prove that it is constrained Pareto optimal. The proofs of these normative properties are based on the following Lemma, which asserts that if a feasible allocation has the property that each agent's consumption stream is co-monotone with aggregate output, then any concave increasing function of aggregate output can be decomposed into the sup-convolution of utility functions $(v^i)_{i\in I}$ such that, for these utility functions, the allocation is the optimal distribution of the aggregate output among the agents.

**Lemma 1.** *Let $\Psi : \mathbb{R}_+ \to \mathbb{R}$ be a concave, increasing, differentiable function satisfying the Inada condition. Let $\bar{x} = (\bar{x}^1,\dots,\bar{x}^I)$ be $I$ vectors in $\mathbb{R}^S_{++}$ with increasing co-ordinates ( $s > s' \Longrightarrow \bar{x}^i_s > \bar{x}^i_{s'}$) and let $y_s$ be defined by $\sum_{i\in I}\bar{x}^i_s = y_s$, $s = 1,\dots,S$. There exist $I$ concave, increasing,*

consume the production of the firm, and there cannot be unanimity of these agents in favor of a change, even if transfers payments are allowed (see Magill-Quinzii (1996) for a more developed exposition of this stability property).

[11] The model with concentrated initial ownership and diffuse "new" ownership corresponds to the moment where the firm goes public. At this exceptional time in the life of the firm, market value maximization is more likely to be applied than expected utility maximization.

*differentiable functions $v^i : \mathbb{R}_+ \to \mathbb{R}$ satisfying the Inada condition such that*

(i) *for all $y \geq 0$, $\Psi(y) = \max\{\sum_{i \in I} v^i(x^i) | x^i \in \mathbb{R}^+, \ \sum_{i \in I} x^i = y\}$*

(ii) $\Psi(y_s) = \sum_{i \in I} v^i(\bar{x}^i_s)$

**Proof:** see Appendix.

**Proposition 5.** *Under Assumptions FS, IN and EU, a firm-expected-utility equilibrium allocation is firm-k optimal, for all $k \in K$.*

Proof : Let $\bar{U}$ denote the "true" profile of utility functions and let $(\bar{a}, \bar{x}, \bar{P})$ be an FEU equilibrium of the economy $\mathcal{E}(\bar{U}, w_0, \delta, y, p)$, with associated discount factor $\bar{\pi}$ and utility functions $(\Phi^k)_{k \in K}$ for the firms. Suppose $(\bar{a}, \bar{x}, \bar{\pi})$ is not firm $k$-optimal, then there exists an alternative allocation $(a, x) = (a_k, \bar{a}^{-k}, x)$ which is feasible, i.e.

$$\sum_{i \in I} x^i_0 + a_k + \sum_{k' \neq k} \bar{a}_{k'} = \sum_{i \in I} w^i_0, \qquad \sum_{i \in I} x^i_s = Y_s, \quad s = 1, \dots S$$

such that for all admissible utility profiles $U$ in $\mathcal{U}(\bar{x}, \bar{\pi})$ defined by (14)

$$u^i_0(x^i_0) + \sum_{s \in S} p_s(a) u^i_1(x^i_s) > u^i_0(\bar{x}^i_0) + \sum_{s \in S} p_s(\bar{a}) u^i_1(\bar{x}^i_s), \quad i \in I \tag{27}$$

To simplify notation, let $\bar{z} = \sum_{k' \in K} \bar{a}_{k'}$ denote the total investment in the FEU equilibrium and let $z = a_k + \sum_{k' \neq k} \bar{a}_{k'}$ denote the total investment in the improving allocation. For each $i \in I$, let $\widetilde{x}^i$ denote the conditional expectation of $x^i$ given $\Sigma$ defined by

$$\widetilde{x}^i_0 = x^i_0, \quad \widetilde{x}^i_\sigma = \sum_{s \in \sigma} \frac{p_s(a)}{p_\sigma(a)} x^i_s, \quad \widetilde{x}^i_s = \widetilde{x}^i_\sigma, \ s \in \sigma, \quad \forall \, \sigma \in \Sigma$$

$(a, \widetilde{x})$ is feasible and, since each agent is risk averse, for all $U^i$ satisfying Assumption EU, $U^i(\widetilde{x}^i, a) \geq U^i(x^i, a)$. Thus we may assume without loss of generality that $x^i = \widetilde{x}^i$, $i \in I$, so that (27) implies

$$u^i_0(x^i_0) + \sum_{\sigma \in \Sigma} p_\sigma(a) u^i_1(x^i_\sigma) > u^i_0(\bar{x}^i_0) + \sum_{\sigma \in \Sigma} p_\sigma(\bar{a}) u^i_1(\bar{x}^i_\sigma), \quad i \in I$$

for all utility profiles $U$ in $\mathcal{U}(\bar{x}, \bar{\pi})$

Let $Y_0 = \sum_{i \in I} w^i_0$ denote the date 0 aggregate resources of the economy, and let $\Phi_0 : \mathbb{R}_+ \to \mathbb{R}$ be an increasing, concave, continuously differentiable function satisfying the Inada condition such that $\Phi'_0(Y_0 - \bar{z}) = 1$. Applying Lemma 1 (with $S = 1$ and the date 0 allocation $\bar{x}_0$) there exist increasing, concave, differentiable functions $(u^i_0)_{i \in I}$ satisfying the Inada condition, such that

(a) for all $\zeta \in [0, Y_0)$, $\Phi_0(Y_0 - \zeta) = \max\{\sum_{i \in I} u_0^i(x_0^i) \,|\, x_0 \in \mathbb{R}_+^I,\ \sum_{i \in I} x_0^i = Y_0 - \zeta\}$

(b) $\Phi_0(Y_0 - \bar{z}) = \sum_{i \in I} u_0^i(\bar{x}_0^i)$

Since $\Phi_0'(Y_0 - \bar{z}) = 1$, for all $i \in I$, $u_0^{i\prime}(\bar{x}_0^i) = 1$. By Proposition 3 each agent's equilibrium date 1 consumption $\bar{x}_{\mathbf{1}}^i$ is a monotone increasing function of $\sigma$: applying Lemma 1 again, there exist concave, increasing, differentiable functions, $(u_1^i)_{i \in I}$ satisfying the Inada condition such that

(c) for all $\eta \in \mathbb{R}_{++}$, $\Phi^k(\eta) = \max\{\sum_{i \in I} u_1^i(\xi^i) | \xi \in \mathbb{R}_+^I,\ \sum_{i \in I} \xi^i = \eta\}$

(d) $\Phi^k(Y_\sigma) = \sum_{i \in I} u_1^i(\bar{x}_\sigma^i),\ \sigma \in \Sigma$

Since $u_1^{i\prime}(\bar{x}_\sigma^i) = \Phi'(Y_\sigma) = \bar{\pi}_\sigma$ for all $i \in I$ and all $\sigma \in \Sigma$ and $u_0^{i\prime}(\bar{x}_0^i) = 1$, the utility functions $(U^i)_{i \in I} = (u_0^i, u_1^i)_{i \in I}$ lie in $\mathcal{U}(\bar{x}, \bar{\pi})$ so that the inequalities (27) are satisfied for these $I$ functions. Summing these inequalities over the agents and using (a)-(d) gives

$$\Phi_0(Y_0 - z) + \sum_{\sigma \in \Sigma} p_\sigma(a) \Phi^k(Y_\sigma) \geq \sum_{i \in I} u_0^i(x_0^i) + \sum_{\sigma \in \Sigma} p_\sigma(a) \sum_{i \in I} u_1^i(x_\sigma^i) > \Phi_0(Y_0 - \bar{z}) + \sum_{\sigma \in \Sigma} p_\sigma(\bar{a}) \Phi^k(Y_\sigma) \tag{28}$$

Since $\Phi_0$ is concave, $\Phi_0(Y_0 - z) \leq \Phi_0(Y_0 - \bar{z}) + \Phi_0'(Y_0 - \bar{z})(\bar{z} - z)$ and, since $\Phi_0'(Y_0 - \bar{z}) = 1$, it follows that

$$\Phi_0(Y_0 - z) - \Phi_0(Y_0 - \bar{z}) \leq \bar{a}_k - a_k \tag{29}$$

(28) and (29) imply

$$\bar{a}_k - a_k \geq \Phi_0(Y_0 - z) - \Phi_0(Y_0 - \bar{z}) > \sum_{\sigma \in \Sigma} p_\sigma(\bar{a}) \Phi^k(Y_\sigma) - \sum_{s \in S} p_\sigma(a) \Phi^k(Y_\sigma)$$

contradicting the fact that $\bar{a}_k$ maximizes the expected social utility $\tilde{V}^k$ given $\bar{a}^{-k}$. □

To obtain further properties of an FEU equilibrium, assumptions of concavity with respect to the investment of firms must be introduced. The following assumption prevents discontinuities in the choice of investment by firms, which can lead to nonexistence of equilibrium.

**Assumption C$_k$ (concavity in $a_k$):** *For all $\eta \geq \underline{Y}$ and all $a \in \mathbb{R}_+^K$*

$$\int_{\underline{Y}}^{\eta} G(t, a_k, a^{-k}) dt \tag{30}$$

*is concave in $a_k$ and continuous in $a$.*

This assumption was introduced in the principal-agent literature by Jewitt (1988) (in the context of one firm) to ensure that the expected utility of the agent is concave in effort. It has a natural interpretation in terms of decreasing returns to investment. Although our analysis does not require it as an explicit assumption, the natural case to consider is where increasing investment $a_k$ increases the probability of high outcomes for firm $k$. Since the output of firm $k$ is added to the aggregate output of the other firms, this can be formalized as the property that an increase in $a_k$ leads to a second-order stochastic dominant shift in the distribution of aggregate output, which is equivalent to the property that the integral (30) is increasing in $a_k$. Concavity of the function in (30) with respect to $a_k$ is then the assumption of stochastic decreasing returns to scale for second-order stochastic dominance.

**Lemma 2.** *If Assumption $C_k$ is satisfied, then for any concave differentiable utility function $\Phi^k$ the objective function $\widetilde{V}^k(a_k, a^{-k})$ is concave in $a_k$ and continuous in $a$.*

Proof. See Appendix.

By Lemma 2, Assumption $C_k$ implies that firm $k$'s optimal choice of investment is an upper-hemi continuous and convex-valued correspondence and this property is used below to prove existence of an equilibrium. A stonger concavity assumption is required to establish the constrained Pareto optimality of an FEU equilibrium.

**Assumption C (concavity in $a$):** *For all $\eta \geq \underline{Y}$ and all $a \in \mathbb{R}^K_+$*

$$\int_{\underline{Y}}^{\eta} G(t, a)dt$$

*is concave and continuous in $a$.*

Assumption C is stronger than Assumption $C_k$ for each $k \in K$, even if the firms' outputs are independent random variables. Consider an example with two firms, firms 1 and 2, with independent outcomes, each with two possible outcomes, H and L. If $p_1(a_1)$ and $p_2(a_2)$ denote the probabilities of the low outcomes, then Assumption $C_k$, for $k = 1, 2$, requires that the functions $p_1$ and $p_2$ are convex. In the Appendix we show that Assumption C requires the product $p_1(a_1)p_2(a_2)$ to be convex in $(a_1, a_2)$. A sufficient condition is that the functions $p_1$ and $p_2$ are log convex, i.e. $\log(p_k(a_k))$ is convex in $a_k$, for $k = 1, 2$.

**Lemma 3.** *If Assumption C is satisfied, then for any concave differentiable utility function $\Phi$ the function $V(a)$ defined by*

$$V(a) = \sum_{s \in S} p_s(a)\Phi(Y_s) - \sum_k a_k \tag{31}$$

*is concave and continuous in $a$, for $a \in \mathbb{R}_+^K$.*

Proof. See Appendix.

To use Lemma 3 to prove constrained Pareto optimality of an FEU equilibrium we need the additional assumption that firms use the same estimated social utility function $\Phi$.

**Proposition 6.** *If $(\bar{a}, \bar{x}, \bar{P})$ is an FEU equilibrium of an economy $\mathcal{E}(\bar{U}, w_0, \delta, y, p)$ satisfying FS, IN, EU, and C, and if $\Phi^k = \Phi$, for all $k \in K$, then $(\bar{a}, \bar{x}, \bar{\pi})$ is constrained Pareto optimal.*

Proof. Since $(\bar{a}, \bar{x}, \bar{P})$ is an FEU equilibrium, for each $k \in K$, $\bar{a}_k$ maximizes

$$\sum_{s \in S} p_s(a_k, \bar{a}^{-k}) \Phi(Y_s) - a_k$$

Thus the first-order conditions for maximizing $V(a)$ in (31) are satisfied at $\bar{a}$, and since $V$ is concave in $a$, $\bar{a}$ is a global maximum of $V$. Let $\bar{\pi}$ be the associated stochastic discount factor and suppose $(\bar{a}, \bar{x}, \bar{\pi})$ is not CPO. Then there exists an alternative feasible allocation $(a, x)$ such that for all admissible utility profiles $U \in \mathcal{U}(\bar{x}, \bar{\pi})$, (27) is satisfied. Using the same argument as in the proof of Proposition 5, choose $\Phi_0$ such that $\Phi_0'(Y_0 - \sum_{k \in K} \bar{a}_k) = 1$ and use Lemma 1 to construct a profile of utility functions $U = (u_0^i, u_1^i)_{i \in I} \in \mathcal{U}(\bar{x}, \bar{\pi})$ such that $\Phi_0$ is the convolution of $(u_0^i)_{i \in I}$ and $\Phi$ is the convolution of $(u_1^i)_{i \in I}$. Summing (27) over the agents for these utility functions and using the properties of $(\Phi_0, \Phi)$ leads to a contradiction to the maximization of expected social utility $V(a)$ at $\bar{a}$. □

The assumptions needed to establish the constrained Pareto optimality of an FEU equilibrium are strong. Joint concavity of the integral (30) is needed to ensure that maximization with respect to each variable in $V(a)$ implies global maximization. We have not succeeded in finding a generalization of the condition for the two-firm two-outcome case to a sum of independent random variables taking an arbitrary number of values, so it is difficult to appreciate the strength of the assumption of joint concavity. The case which we can solve suggests that the decreasing returns to investment by each firm must be sufficiently strong. The assumption that $\Phi^k$ is the same for all firms may appear restrictive since there may be no obvious reason why all firms should "fill in" the missing information on marginal utilities in the same way. However if there is a conventional way of estimating a function which is only known at a finite number of points—for example by linear interpolation—then all firms may use the same function $\Phi$ to estimate social welfare.

Simplifying the proof of Proposition 6—using the "true" utilities $(\bar{u}_0^i, \bar{u}_0^i)$ instead of using Lemma

1 to find admissible utilities—leads to the Pareto optimality of a Strong FEU equilibrium under Assumption C.

**Proposition 7.** *If $(\bar{a}, \bar{x}, \bar{P})$ is a Strong FEU equilibrium of an economy $\mathcal{E}(U, w_0, \delta, y, p)$ satisfying FS, IN, EU, and C, then $(\bar{a}, \bar{x})$ is Pareto optimal.*

**Existence of Equilibrium.** In an FEU equilibrium the "valuation" of a firm's investment is made in two distinct ways. Its choice of investment $a_k$ is based on the expected social utility criterion

$$\widetilde{V}^k(a_k, a^{-k}) = \sum_{s \in S} p_s(a_k, a^{-k}) \Phi^k(y^k_{s_k} + Y^{-k}_s) - a_k$$

In the budget equations of the investors, where the present value of a firm's profit is one of the sources of income for its shareholders, the valuation of a firm's investment enters through the market value

$$M^k(a_k, a^{-k}) = \sum_{s \in S} p_s(a_k, a^{-k}) \pi_s y^k_{s_k} - a_k$$

These two valuations may not always be compatible, causing potential problems for existence of an FEU equilibrium. For, while investors as a group benefit more from the increase in expected utility obtained by maximizing the criterion $\widetilde{V}^k$ than they give up by not maximizing market value, maximizing $\widetilde{V}^k$ may sometimes lead the market value $M^k$ to become negative. When this happens, an agent with a small initial endowment $w^i_0$ and a relatively large ownership share of the firm may end up with a negative income, and this is incompatible with equilibrium.

There is a natural way of resolving this potential incompatibility. For the reason why agents are better off when firm $k$ maximizes $\widetilde{V}^k$ rather than $M^k$ is that its investment has a positive externality on all agents via the probability term $p(a)$ in their expected utilities. If this external effect is strong enough to justify "overinvestment" to the point where the firm's market value becomes negative, then it seems natural to compensate the initial owners by making transfers to the firm which ensure that its market value remains non-negative. Such transfers would need to be financed by taxes: for simplicity it is assumed that they are financed by a uniform tax rate on the endowments $w^i_0$ of all agents.[12] This suggests modifying the concept of an FEU equilibrium by introducing a tax rate $\tau$ on agents' initial incomes and a vector of transfers $t = (t_k)_{k \in K}$ to the firms: for economies for which transfers are not needed, the resulting concept is just an FEU equilibrium.

[12]The transfers could be chosen to ensure a minimum rate of return on the firm's equity and/or the taxes could depend on the level of income: under reasonable conditions this will not create difficulties for the existence of an equilibrium.

When the tax rate is $\tau$ on initial income and transfers to the firms are $t$, the budget set of an agent becomes

$$B(P, w_0^i, \delta^i, a; \tau, t) = \left\{ x^i \in \mathbb{R}_+^{S+1} \;\middle|\; Px^i = w_0^i(1-\tau) + \sum_{k \in K} \delta_k^i (P_{\mathbf{1}} y^k - a^k + t^k) \right\}$$

This leads to the definition of an FEU equilibrium with transfers.

**Definition 5.** *Let $\mathcal{E}(U, w_0, \delta, y, p)$ be an economy satisfying Assumptions FS, EU, and IN. $(\bar{a}, \bar{x}, \bar{P}; \bar{\tau}, \bar{t})$ is an firm-expected-utility equilibrium with transfers (FEUT equilibrium) if*

(i) *each agent $i \in I$ chooses $\bar{x}^i$ which maximizes $U^i(x^i, \bar{a})$ in the budget set $B(\bar{P}, w_0^i, \delta^i, \bar{a};\, \bar{\tau}, \bar{t})$*

(ii) *each firm $k \in K$ chooses the investment $a_k$ which maximizes its estimated expected social utility $\widetilde{V}^k(a_k, \bar{a}^{-k})$.*

(iii) $\sum_{i \in I} \bar{x}_0^i + \sum_{k \in K} \bar{a}_k = \sum_{i \in I} w_0^i, \qquad \sum_{i \in I} \bar{x}_s^i = Y_s,\ s \in S$

(iv) $\bar{\tau} \sum_{i \in I} w_0^i = \sum_{k \in K} \bar{t}_k$

Adding taxes and transfers to a Strong FEU equilibrium leads in the same way to a Strong FEUT equilibrium. Introducing taxes and transfers in this way, to cover cases where firms can potentially have negative market values, allows us to establish existence of equilibrium.

**Proposition 8.** *If $\mathcal{E}(U, w_0, \delta, y, p)$ is an economy satisfying Assumptions FS, EU, IN and $C_k$ for all $k \in K$, then there exists an FEUT and a Strong FEUT equilibrium.*

Proof: see Appendix.

**Example.** Consider an economy with $I$ consumers and $K = 2$ firms, where each firm has two possible outputs $y^1 = (y_L^1, y_H^1) = (400, 650)$, $y^2 = (y_L^2, y_H^2) = (500, 700)$ so that the possible aggregate outcomes are $(Y_1, Y_2, Y_3, Y_4) = (900, 1100, 1150, 1350)$. Firms' outcomes are independent, and firm $k$'s ability to shift probability away from its low outcome by investment is given by $p_k(a_k) = e^{-\nu_k a_k}$, with $\nu_1 = \nu_2 = 0.05$. Since the functions $p_k$ are log-convex, Assumption C is satisfied. Suppose agents have identical homothetic preferences

$$U(x^i, a) = \log(x_0^i) + \beta \sum_{s \in S} p_s(a) \log(x_s^i), \quad i = 1, 2$$

and initial endowments $(w_0^i, \delta^i)$ such that $\sum_{i \in I} w_0^i = 1000$ and $\sum_{i \in I} \delta_k^i = 1, k = 1, 2$. This is a representative agent economy, and if $(a_1, a_2)$ is the investment of the firms, then the prices in the consumption equilibrium with fixed investment are $P_\sigma = p_\sigma(a)(\sum_{i\in I} w_0^i - a_1 - a_2)/Y_\sigma$, $\sigma = 1,\ldots,4$. Since prices are independent of the distribution of income and each agent has a consumption stream proportional to the aggregate output $(\sum_{i\in I} w_0^i - a_1 - a_2, Y_1, \ldots, Y_4)$, in what follows we omit the consumption component $\bar{x} = (\bar{x}^i)_{i\in I}$ of an equilibrium. The utility function $\Phi_{\bar{x}_0}$ in (5) is, up to a constant, given by $\Phi_{\bar{x}_0}(\eta) = (\sum_{i\in I} w_0^i - \bar{a}_1 - \bar{a}_2)\log(\eta)$, so that $(a_1^*, a_2^*)$ is a Strong FEU equilibrium if

$$a_k^* = \arg\max(\sum_{i\in I} w_0^i - a_1^* - a_2^*) \sum_{\sigma=1}^{4} p_\sigma(a_k, a^{*-k}) \log(Y_\sigma) - a_k, \quad k = 1, 2$$

For the choice of parameters given above

$$a^* = (55, 46.8)$$

so that approximately 10% of date 0 income is devoted to investment. If the agents' utility functions are not known and both firms use the same linear interpolation (21), then $(\bar{a}_1, \bar{a}_2)$ is an FEU equilibrium choice of investment if

$$\bar{a}_k = \arg\max \sum_{\sigma=1}^{4} p_\sigma(a_k, \bar{a}^{-k}) \Phi^k(Y_\sigma) - a_k, \quad k = 1, 2$$

with $\Phi^1 = \Phi^2$, $\Phi^1(Y_1) = 0$, and $\Phi^1(Y_{\sigma-1}) = \Phi^1(Y_\sigma) + \frac{1}{2}(\sum_{i\in I} w_0^i - \bar{a}_1 - \bar{a}_2)(\frac{1}{Y_\sigma} + \frac{1}{Y_{\sigma+1}})(Y_{\sigma+1} - Y_\sigma)$. The FEU equilibrium choice of investment is

$$\bar{a} = (55.1, 46.9)$$

which only marginally exceeds the first-best investment level.[13]

All the FEU equilibria correspond to investments $(\tilde{a}_1, \tilde{a}_2)$ which are optimal given estimated utility functions $(\Phi^k, k = 1, 2)$ satisfying

$$\frac{\sum_{i\in I} w_0^i - \tilde{a}_1 - \tilde{a}_2}{Y_{\sigma+1}}(Y_{\sigma+1} - Y_\sigma) < \Phi^k(Y_{\sigma+1}) - \Phi^k(Y_\sigma) < \frac{\sum_{i\in I} w_0^i - \tilde{a}_1 - \tilde{a}_2}{Y_\sigma}(Y_{\sigma+1} - Y_\sigma)$$

Suppose firm 1 adopts the "low estimate" $\Phi^1$ defined by $\Phi^1(Y_1) = 0$, $\Phi^1(Y_{\sigma+1}) = \Phi^1(Y_\sigma) + (\sum_{i\in I} w_0^i - \tilde{a}_1 - \tilde{a}_2)(0.01\frac{1}{Y_\sigma} + 0.99\frac{1}{Y_{\sigma+1}})(Y_{\sigma+1} - Y_\sigma)$, while firm 2 adopts the "high estimate" $\Phi^2$ defined by permuting the weights 0.01 and 0.99, then the FEU investment is

$$\tilde{a} = (52.5, 49.8)$$

[13] Since $\pi(\eta) = (\sum_{i\in I} w_0^i - \bar{a}_1 - \bar{a}_2)/\eta$ is a convex function of $\eta$, $\phi^k(\eta) > \pi(\eta)$ for $Y_\sigma < \eta < Y_{\sigma+1}$. The function chosen by the firms slightly exaggerates the increases in social welfare so that the investment is marginally too high.

Firm 1 underestimates the increases in utility and thus invests less than in the first best, while firm 2 overestimates the increases in utility and overinvests relative to the first best. Is the investment $\tilde{a}$ constrained efficient? To answer this question we check the value of the partial derivatives $\frac{\partial}{\partial a_k}\sum_{\sigma=1}^{4} p(a)\Phi_{\lambda,\mu,\nu}(Y_\sigma)$, $k = 1, 2$, where the functions $\Phi_{\lambda,\mu,\nu}$ are such that $\Phi_{\lambda,\mu,\nu}(Y_1) = 0$ and

$$\Phi_{\lambda,\mu,\nu}(Y_2) - \Phi_{\lambda,\mu,\nu}(Y_1) = \left(\frac{\lambda}{Y_1} + \frac{1-\lambda}{Y_2}\right)(Y_2 - Y_1)(\sum_{i\in I} w_0^i - \tilde{a}_1 - \tilde{a}_2)$$
$$\Phi_{\lambda,\mu,\nu}(Y_3) - \Phi_{\lambda,\mu,\nu}(Y_2) = \left(\frac{\mu}{Y_2} + \frac{1-\mu}{Y_3}\right)(Y_3 - Y_2)(\sum_{i\in I} w_0^i - \tilde{a}_1 - \tilde{a}_2)$$
$$\Phi_{\lambda,\mu,\nu}(Y_4) - \Phi_{\lambda,\mu,\nu}(Y_3) = \left(\frac{\nu}{Y_3} + \frac{1-\nu}{Y_4}\right)(Y_4 - Y_3)(\sum_{i\in I} w_0^i - \tilde{a}_1 - \tilde{a}_2)$$

for a grid of values $(\lambda, \mu, \nu) \in [0,\ 1]^3$, and find that in all cases $\frac{\partial}{\partial a_1}\sum_{\sigma=1}^{4} p(\tilde{a})\Phi_{\lambda,\mu,\nu}(Y_\sigma) > \frac{\partial}{\partial a_2}\sum_{\sigma=1}^{4} p(\tilde{a})\Phi_{\lambda,\mu,\nu}(Y_\sigma)$ so that a marginal transfer of investment from firm 2 to firm 1 would improve social welfare for any admissible profile of utility functions. Thus the assumption that firms use the same estimate $\Phi^k$ of the social utility function seems to be an unavoidable condition for the constrained Pareto optimality result of Proposition 6. If instead firm 1 adopts the "high estimate" and firm 2 uses the "low estimate" then the equilibrium investment is $\tilde{a} = (57.6, 44)$. This suggests that even though there is a continuum of FEU equilibria, the equilibrium investment levels lie in the relatively restricted intervals [52, 58] for firm 1 and [44, 50] for firm 2. Since the market value of each firm is positive in each of these equilibria, no transfers are required.

The difference between investment in the first-best equilibrium and in a market-value maximizing equilibrium is much more pronounced. If firms behave "competitively" and take the discount factor $\bar{\pi}$ as given, then $(\hat{a}_1, \hat{a}_2)$ is a market-value maximizing choice of investment if

$$\hat{a}_k = \arg\max(\sum_{i\in I} w_0^i - \hat{a}_1 - \hat{a}_2)\sum_{\sigma=1}^{4} p_\sigma(a_k, \hat{a}^{-k})\frac{y_\sigma^k}{Y_\sigma} - a_k, \quad k = 1, 2$$

$(\hat{a}_1, \hat{a}_2) = (36.9, 24.7)$, so that $\hat{a}_1$ is about 1/2 and $\hat{a}_2$ about 1/3 of the first-best investment. If firms behave "non-competitively" and explicitly take into account the effect of their investment on $\bar{\pi}$, then the market-value maximizing investment becomes $(\bar{\bar{a}}_1, \bar{\bar{a}}_2) = (25.8, 12.8)$: since increasing investment reduces $\bar{\pi}$, their investment is reduced even further.

## 7. Conclusion

In this paper we have studied a probability approach to modeling equilibrium in a production economy under uncertainty in the setting of a one good economy in which investment by firms influences the probability distributions of their outcomes. We found that in such a model, the objective function that each firm "should" maximize involves explicitly taking into account the effect of its investment on social welfare. To understand the role that prices play in such an economy, we considered a setting where firms do not know the agents' utility functions. We showed how the information contained in prices can be used to estimate the social utility function and to construct objective functions which lead to second-best choices of investment.

In the simple calculated example of the previous section in which each firm contributes about half the risk of the economy, the difference in investment between the first-best and the market-value maximizing investment is substantial. However we have shown that if a firm is marginal its objective is not far from market value. It seems likely that in a calibrated model with a larger number of firms of more realistic sizes, the difference between first or second-best and market-value maximizing investment will be much smaller. It should be remembered however that this paper is based on the strong assumption that markets span all the uncertainty in outcomes and that agents have perfectly diversifiable risks. Intuitively, if agents have risks which are directly linked to the random outcomes of the firms and if these risks are not perfectly diversifiable—labor risks are a prime example—then the results of this paper suggest that firms should explicitly take into account the risk consequences of their investment decisions for consumers, and that this could lead to investment which differs substantially from market-value maximizing investment. In order to study these and related questions, the model in this paper needs to be generalized to an environment in which agents face nondiversifiable risks. We plan to address these issues in future research.

## Appendix

**Proof of Proposition 4.** We begin by showing that Assumption FOSD$_k$ implies $\Sigma_- = \emptyset$. For any subset $E$ of the outcome space $S$, let $\mathbb{P}(E, a) = \sum_{s \in E} p_s(a)$ denote the probability of event $E$. For any firm $k$, we can decompose the aggregate output $Y$ at date 1 into the sum of firm $k$'s output and the total output of the other firms: $Y = y^k + Y^{-k}$. In view of NE, the upper-cumulative function of aggregate output can be written as

$$\begin{aligned} G(\eta, a) &= \mathbb{P}\left[(y^k + Y^{-k}) > \eta;\, a\right] = \sum_{s^{-k}} \mathbb{P}\left[y^k > \eta - Y^{-k}_{s^{-k}},\ y^{-k} = y^{-k}_{s^{-k}};\, a\right] \\ &= \sum_{s^{-k}} p(s^{-k}, a^{-k}) G^k(\eta - Y^{-k}_{s^{-k}};\, s^{-k},\, a_k,\, a^{-k}) \end{aligned}$$

where $G^k$ is the upper-cumulative distribution function of $y^k$ with the conditional probability $p_{s_k}(a_k|s^{-k}, a^{-k})$. By FOSD$_k$, $G^k$ is an increasing function of $a_k$ and hence $G$, as a positive linear combination of these functions, is an increasing function of $a_k$. Thus $\Sigma_- = \emptyset$ and

$$\underline{W_{a_k}} = \sum_{\Sigma_+} G_{a_k}(Y_\sigma, \bar{a})\bar{\pi}_{\sigma+1}(Y_{\sigma+1} - Y_\sigma)$$

Consider the piecewise linear function $\underline{\Phi}$ of aggregate output induced by the stochastic discount factor $\bar{\pi}$

$$\underline{\Phi}(\eta) = \begin{cases} 0 & \text{if} \quad \eta \leq \underline{Y} \\ \underline{\Phi}(Y_\sigma) + \bar{\pi}_{\sigma+1}(\eta - Y_\sigma) & \text{if} \quad \eta \in [Y_\sigma,\ Y_{\sigma+1}],\ \sigma = 1, \ldots, \Sigma - 1 \\ \underline{\Phi}(\overline{Y}) & \text{if} \quad \eta \geq \overline{Y} \end{cases}$$

By construction

$$\underline{\Phi}(Y_{\sigma+1}) - \underline{\Phi}(Y_\sigma) = \bar{\pi}_{\sigma+1}(Y_{\sigma+1} - Y_\sigma), \quad \sigma = 1, \ldots, \Sigma - 1 \tag{32}$$

so that $\underline{W_{a_k}}$ can be written (using the (IP) relation)

$$\underline{W_{a_k}} = \sum_{\sigma=1}^{\Sigma-1} G_{a_k}\,(Y_\sigma, \bar{a})(\underline{\Phi}(Y_{\sigma+1}) - \underline{\Phi}(Y_\sigma)) = \frac{\partial}{\partial a_k} \sum_{\sigma\in\Sigma} p_\sigma(\bar{a})\underline{\Phi}(Y_\sigma) = \frac{\partial}{\partial a_k} \sum_{s\in S} p_s(\bar{a})\underline{\Phi}(Y_s)$$

since $p_\sigma(\bar{a}) = \sum_{s\in\sigma} p_s(\bar{a})$. Let $R^k(a_k) = \sum_{s\in S} p_s(a_k, \bar{a}^{-k})\bar{\pi}_s y^k_{s_k}$ denote the present value of firm $k$'s date 1 revenue. Since $\bar{a}_k > 0$ maximizes the market value $M(a_k, \bar{a}^{-k})$ the first-order condition $R^k_{a_k} = 1$ must hold. As a result, showing that $\underline{W_{a_k}} > 1$ is equivalent to showing that

$$\begin{aligned} \underline{W_{a_k}} - R^k_{a_k} &= \frac{\partial}{\partial a_k} \sum_{s\in S} p_s(\bar{a})(\underline{\Phi}(Y_s) - \bar{\pi}_s y^k_{s_k}) \\ &= \frac{\partial}{\partial a_k} \sum_{s^{-k}} p_{s^{-k}}(\bar{a}^{-k}) \sum_{s_k} p_{s_k}(\bar{a}^k \mid s^{-k}, \bar{a}^{-k}) \left(\underline{\Phi}(y^k_{s_k} + Y^{-k}_{s^{-k}}) - \bar{\pi}_{(s_k, s^{-k})} y^k_{s_k}\right) \end{aligned} \tag{33}$$

is positive. When $Y^{-k}_{s^{-k}}$ is fixed, two consecutive values of $y^k$ correspond to two consecutive values of the aggregate output $Y$. By (32)

$$\underline{\Phi}(y^k_{s_{k+1}} + Y^{-k}_{s^{-k}}) - \bar{\pi}_{(s_{k+1}, s^{-k})} y^k_{s_{k+1}} = \underline{\Phi}(y^k_{s_k} + Y^{-k}_{s^{-k}}) - \bar{\pi}_{(s_{k+1}, s^{-k})} y^k_{s_k}$$

and by the antimonotonicity of $\pi$ in Proposition 1, $\bar{\pi}_{(s_{k+1}, s^{-k})} < \bar{\pi}_{(s_k, s^{-k})}$, so that

$$\underline{\Phi}(y^k_{s_{k+1}} + Y^{-k}_{s^{-k}}) - \bar{\pi}_{(s_{k+1}, s^{-k})} y^k_{s_{k+1}} > \underline{\Phi}(y^k_{s_k} + Y^{-k}_{s^{-k}}) - \bar{\pi}_{(s_k, s^{-k})} y^k_{s_k}$$

Thus for $s^{-k}$ fixed, the function $s_k \to \underline{\Phi}(y^k_{s_k} + Y^{-k}_{s^{-k}}) - \bar{\pi}_{(s_k, s^{-k})} y^k_{s_k}$ is increasing in $s_k$. The first-order-stochastic-dominance property FOSD$_k$ then implies

$$\frac{\partial}{\partial a_k} \sum_{s_k \in S_k} p_{s_k}(\bar{a}^k \mid s^{-k}, \bar{a}^{-k}) \left(\underline{\Phi}(y^k_{s_k} + Y^{-k}_{s^{-k}}) - \bar{\pi}_{(s_k, s^{-k})} y^k_{s_k}\right) > 0$$

so that (33) is positive and the proof is complete. □

**Proof of Lemma 1.** Let $h^i : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$h^i(y) = \begin{cases} \dfrac{y}{y_1}\bar{x}^i_1 & \text{if} \quad y \in [0, y_1] \\ \dfrac{y_{s+1} - y}{y_{s+1} - y_s}\bar{x}^i_s + \dfrac{y - y_s}{y_{s+1} - y_s}\bar{x}^i_{s+1} & \text{if} \quad y \in [y_s, y_{s+1}],\ s = 1, \ldots, S-1 \\ \dfrac{y}{y_S}\bar{x}^i_S & \text{if} \quad y \geq y_S \end{cases}$$

$h^i$ is continuous, increasing, differentiable on the intervals $(0, y_1)$, $(y_s, y_{s+1})$, $s = 1, \ldots, S-1$, and $(y_S, \infty)$, and satisfies

$$h^i(y_s) = \bar{x}^i_s, \quad \forall\ s \in S, \quad h^{i\prime}(y) = \begin{cases} \dfrac{\bar{x}^i_1}{y_1} & \text{if} \quad y \in (0, y_1) \\ \dfrac{\bar{x}^i_{s+1} - \bar{x}^i_s}{y_{s+1} - y_s} & \text{if} \quad y \in (y_s, y_{s+1}),\ s = 1, \ldots, S-1 \\ \dfrac{\bar{x}^i_S}{y_S} & \text{if} \quad y > y_S \end{cases}$$

Note that

$$\sum_{i \in I} h^i(y) = y, \quad \sum_{i \in I} h^{i\prime}(y) = 1, \quad y \in \mathbb{R}_+$$

since $\sum_{i \in I} h^{i\prime}(y)$ can be extended by continuity to $\mathbb{R}_+$.

Since $h^i$ is increasing, it can be inverted; let $(h^i)^{-1}$ denote the inverse function which is also continuous, increasing, and differentiable on the intervals $(0, y_1)$, $(y_s, y_{s+1})$, $s = 1, \ldots, S-1$, $(y_S, \infty)$. For $i = 1, \ldots, I$, define a function $v^i$ by

$$v^i(\xi^i) = \int_{\bar{x}^i_1}^{\xi^i} \Psi'((h^i)^{-1}(t))dt \tag{34}$$

We assume w.l.o.g. that $\Psi(y_1) = 0$, so that we choose $v^i$ such that $v^i(\bar{x}^i_1) = 0$. Note that since $\Psi'$ is decreasing and $(h^i)^{-1}$ is increasing, $v^i$ is concave. Consider the maximum problem

$$\max \left\{ \sum_{i \in I} v^i(\xi^i) \,\middle|\, \xi^i \in \mathbb{R}_+,\ \sum_{i \in I} \xi^i = y \right\} \tag{35}$$

Since $\sum_{i \in I} h^i(y) = y$ and for all $i$, $v^{i\prime}(h^i(y)) = \Psi'((h^i)^{-1}(h^i(y))) = \Psi'(y)$, i.e. the derivatives of the functions $v^i$ are equalized at $h^i(y)$, the solution to the maximum problem is $\xi^i = h^i(y)$ for all $i$, and thus the value of the maximum is $\sum_{i \in I} v^i(h^i(y))$. To show that $\sum_{i \in I} v^i(h^i(y)) = \Psi(y)$, note that, by decomposing the integral in (34) into a sum of integrals over the intervals $(\bar{x}^i_s, \bar{x}^i_{s+1})$ and performing the change of variable $z = (h^i)^{-1}(t) \Longleftrightarrow h^i(z) = t \Longrightarrow h^{i\prime}(z)dz = dt$ on each interval, one obtains

$$v^i(\xi^i) = \int_{y_1}^{y} \Psi'(z) h^{i\prime}(z)dz, \qquad \text{where } y = (h^i)^{-1}(\xi^i)$$

Thus

$$\sum_{i\in I} v^i(h^i(y)) = \int_{y_1}^{y} \Psi'(z) \sum_{i\in I} h^{i\prime}(z)dz = \Psi(y)$$

since $\sum_{i\in I} h^{i\prime}(z) = 1$. Since $\bar{x}_s^i = h^i(y_s)$, $\sum_{i\in I} v^i(\bar{x}_s^i) = \Psi(y_s)$. Also, since by (34) $v^{i\prime}(\xi^i) = \Psi'((h^i)^{-1}(\xi^i))$, and since by construction of $h^i$, $(h^i)^{-1}(\xi^i) \to 0$ when $\xi^i \to 0$, $v^i$ inherits the Inada property from $\Psi$, and the functions $v^i$ have the properties asserted in Lemma 1. □

**Proof of Lemmas 2 and 3.** Let $V(\alpha, \beta) = \sum_{s\in S} p_s(\alpha, \beta)\Phi(Y_s) - \alpha\mathbf{1}$ where $\Phi : \mathbb{R} \to \mathbb{R}$ is a concave, increasing, differentiable function, $p_s(\alpha, \beta)$ is a probability distribution depending on parameters $\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}^m$, $(Y_s)_{s\in S}$ is a real valued random variable with minimum value $\underline{Y}$ and maximum value $\overline{Y}$, and $\mathbf{1}$ is the unit vector in $\mathbb{R}^n$. Let $\Sigma$ be an index set for the values of $Y$ such that $\sigma > \sigma' \Longrightarrow Y_\sigma > Y_{\sigma'}$, and for $\sigma \in \Sigma$, let $p_\sigma(\alpha, \beta) = \sum_{\{s\in S | Y_s = Y_\sigma\}} p_s(\alpha, \beta)$, so that

$$V(\alpha, \beta) = \sum_{\sigma\in\Sigma} p_\sigma(\alpha, \beta)\Phi(Y_\sigma) - \alpha\mathbf{1} \tag{36}$$

If $G$ denotes the upper cumulative distribution function of $Y$ and $\Phi(\underline{Y}) = 0$, then (IP) implies that

$$V(\alpha, \beta) = \sum_{\sigma=1}^{\Sigma-1} G(Y_\sigma, \alpha, \beta) \int_{Y_\sigma}^{Y_{\sigma+1}} \phi(t)dt = \int_{\underline{Y}}^{\overline{Y}} G(t, \alpha, \beta)\phi(t)dt$$

where $\phi$ denotes the positive, continuous, decreasing derivative of $\Phi$. Being decreasing it is differentiable almost everywhere and $\phi'(t) < 0$. Let $H(t, \alpha, \beta)$ denote the integral of $G$ defined by $H(t, \alpha, \beta) = \displaystyle\int_{\underline{Y}}^{t} G(\tau, \alpha, \beta)d\tau$. Integrating (36) by parts leads to

$$V(\alpha, \beta) = \phi(\overline{Y})H(\overline{Y}, \alpha, \beta) - \int_{\underline{Y}}^{\overline{Y}} H(t, \alpha, \beta)\phi'(t)dt \tag{37}$$

Since $H$ is concave in $\alpha$, $\phi(\overline{Y}) > 0$, $-\phi'(t) > 0$, and the sum of concave functions is concave, $V$ is concave in $\alpha$. It is also clear from (37) that if $H$ is continuous in $(\alpha, \beta)$, $V$ inherits this property. Lemma 2 follows with $n = 1, m = K - 1$, $\alpha = a_k$ and $\beta = a^{-k}$ while Lemma 3 follows with $n = K, m = 0$ and $\alpha = a$. □

**Assumption C in the Two-Firm-Two-Outcome Case.** Suppose each of two firms $(k = 1, 2)$ can produce either $y_L^k$ with probability $p_k(a_k)$ or $y_H^k$ with probability $1 - p_k(a_k)$, the outcomes of the two firms being independent. Without loss of generality we assume that

$$y_L^1 + y_L^2 < y_H^1 + y_L^2 \leq y_L^1 + y_H^2 < y_H^1 + y_H^2$$

It is easy to see that Assumption C$_k$ is equivalent to $p_k$ being convex, for $k = 1, 2$ (with two outcomes there is no difference between first-order and second-order stochastic dominance). Let $F(\eta, a_1, a_2)$ denote the (lower) cumulative distribution function of $Y = y^1 + y^2$. Assumption C is equivalent to the convexity of the function

$$M(\eta, a) = \int_{\underline{Y}}^{\eta} F(t, a)dt$$

in $a = (a_1, a_2)$ for all $\eta \in [\underline{Y}\ \overline{Y}]$. Let $I_1 = [y_L^1 + y_L^2 \quad y_H^1 + y_L^2]$, $I_2 = [y_H^1 + y_L^2 \quad y_L^1 + y_H^2]$, and $I_3 = [y_L^1 + y_H^2 \quad y_H^1 + y_H^2]$. Simple calculations lead to

$$M(\eta, a) = \begin{cases} (\eta - y_L^1 - y_L^2)\, p_1(a_1)\, p_2(a_2) & \text{if} \quad \eta \in I_1 \\ (y_H^1 - y_L^1)\, p_1(a_1)\, p_2(a_2) + (\eta - y_H^1 - y_L^2)\, p_2(a_2) & \text{if} \quad \eta \in I_2 \\ \begin{array}{r} (y_H^1 - y_L^1)\, p_1(a_1)\, p_2(a_2) + (y_L^1 + y_H^2 - y_H^1 - y_L^2)\, p_2(a_2) \\ +(\eta - y_L^1 - y_H^2)(p(a_2) + p(a_1)(1 - p_2(a_2))) \end{array} & \text{if} \quad \eta \in I_3 \end{cases}$$

Since the coefficient of $p(a_1)p(a_2)$ in the expression for $M$ when $\eta \in I_3$ is

$$(y_H^1 - y_L^1) - (\eta - y_L^1 - y_H^2) \leq (y_H^1 - y_L^1) - (y_H^1 + y_H^2 - y_L^1 - y_H^2) \leq 0$$

convexity of $p_1(a_1)p_2(a_2)$ in $(a_1, a_2)$ is necessary and sufficient for the convexity of $M(\eta, a)$ in $(a_1, a_2)$ for all $\eta \in [\,\underline{Y}\ \overline{Y}\,]$.

If $\log(p(a_1)$ and $\log(p(a_2)$ are convex, then $\log(p(a_1)) + \log(p(a_1)) = \log(p(a_1)p(a_2))$ is convex, and $p(a_1)p(a_2)$ is convex. □

**Proof of Proposition 8.** We prove existence of an FEUT equilibrium: a simple adaptation of the proof leads to the existence of a Strong FEUT equilibrium. To prove that there exists $(\bar{a}, \bar{x}, \bar{\pi}, \bar{\tau}, \bar{t})$ satisfying (i)-(iv), we use the Negishi approach, showing that a correspondence $\Gamma$, $(\alpha, a, t) \xrightarrow{\Gamma} (\tilde{\alpha}, \tilde{a}, \tilde{t})$ from a compact set (defined below) into itself has a fixed point which is an FEUT equilibrium. $\alpha = (\alpha_i)_{i \in I}$ is the vector of weights associated with a Pareto optimal distribution of output, $a = (a_k)_{k \in K}$ is the vector of investments by firms and $t = (t_k)_{k \in K}$ is the vector of transfers to firms. The domain of the correspondence $\Gamma$ is $\mathbb{D}_\epsilon = \Delta^I \times D_\epsilon^K \times D_\epsilon^K$ where

$$\Delta^I = \left\{ \alpha \in \mathbb{R}_+^I \,\middle|\, \sum_{i \in I} \alpha_i = 1 \right\}, \qquad D_\epsilon^K = \left\{ a \in \mathbb{R}_+^K \,\middle|\, \sum_{k \in K} a_k \leq \sum_{i \in I} w_0^i - \epsilon \right\}$$

with $0 < \epsilon < \sum_{i \in I} w_0^i$: thus $\Delta^I$ is the simplex in $\mathbb{R}^I$ and $D_\epsilon^K$ is the set of investments by the firms which assure at least $\epsilon$ units of consumption for the agents at date 0. We will show that when $\epsilon$ is sufficiently small, a fixed point of $\Gamma$ is an equilibrium.

For $(\alpha, a, t) \in \mathbb{D}_\epsilon$, let $x(\alpha, a, t)$ be the solution of the problem of maximizing social welfare

$$x(\alpha, a, t) = \text{argmax}_{x \in \mathbb{R}_+^{(S+1)I}} \left\{ \sum_{i \in I} \alpha_i U^i(x^i, a) \;\middle|\; \sum_{i \in I} x_0^i + \sum_{k \in K} a_k = \sum_{i \in I} w_0^i, \quad \sum_{i \in I} x_s^i = Y_s, \; s \in S \right\} \tag{38}$$

By Assumption EU, $x(\alpha, a, t)$ and the associated supporting price $(1, P_{\mathbf{1}}(\alpha, a, t))$ (where $P_0$ is normalized to 1) is unique and the map $(\alpha, a, t) \to (x(\alpha, a, t), P(\alpha, a, t))$ is continuous. Any Pareto optimal allocation of $(\sum_{i \in I} w_0^i - \sum_{k \in K} a_k, \, Y)$ satisfies the monotonicity properties of Proposition 1, so that if $\pi_s(\alpha, a, t)$ is defined by $P_s(\alpha, a, t) = p_s(a)\pi_s(\alpha, a, t)$ for $s \in S$ , then $\pi$ satisfies $\pi_s(\alpha, a, t) = \pi_{s'}(\alpha, a, t)$, for all $s, s' \in \sigma$, where $\sigma \in \Sigma$ is such that $Y_s = Y_\sigma$, for all $s \in \sigma$.

Let

$$\tau(\alpha, a, t) = \frac{\sum_{k \in K} t_k}{\sum_{i \in I} w_0^i} = \frac{\sum_{k \in K} t_k}{\sum_{i \in I} w_0^i} < 1$$

and let $\xi(\alpha, a, t)$ be the excess expenditure map

$$\xi_i(\alpha, a, t) = w_0^i(1 - \tau) + \sum_{k \in K} \delta_k^i (P_{\mathbf{1}} y^k - a_k - t_k) - P x^i, \quad i \in I$$

where, for simplicity, the argument $(\alpha, a, t)$ of the functions $x(\alpha, a, t), P(\alpha, a, t), \tau(\alpha, a, t)$ has been omitted. By feasibility of $x(\alpha, a, t)$ and the definition of $\tau(\alpha, a, t)$, $\sum_{i \in I} \xi_i(\alpha, a, t) = 0$.
The first component $\tilde{\alpha}$ of the correspondence $\Gamma$ is defined by

$$\tilde{\alpha}_i(\alpha, a, t) = \frac{\alpha_i + \max\{\xi_i(\alpha, a, t), 0\}}{1 + \sum_{j=1}^J \max\{\xi_j(\alpha, a, t), 0\}}, \qquad i \in I$$

Since $\pi_s(\alpha, a, t)$ is $\Sigma$-measurable, let $\pi_\sigma(\alpha, a, t) = \pi_s(\alpha, a, t)$ for $s \in \sigma$, and let $\phi_{(\alpha,a,t)}$ be the linear interpolation

$$\phi_{(\alpha,a,t)}(Y) = \pi_\sigma + \frac{Y - Y_\sigma}{Y_{\sigma+1} - Y_\sigma}(\pi_{\sigma+1} - \pi_\sigma), \quad \text{for } Y \in [Y_\sigma, Y_{\sigma+1}], \; \forall \sigma \leq \Sigma - 1$$

where again the argument $(\alpha, a, t)$ of $\pi(\alpha, a, t)$ has been omitted. $\phi_{(\alpha,a,t)}(Y)$ and its integral $\Phi_{(\alpha,a,t)}(Y)$ are continuous functions of $(\alpha, a, t, Y)$ for $(\alpha, a, t) \in \mathbb{D}_\epsilon$ and $Y \in [\underline{Y}, \overline{Y}]$, where $\Phi_{(\alpha,a,t)}(Y)$ is defined by

$$\Phi_{(\alpha,a,t)}(Y) = \int_{\underline{Y}}^{Y} \phi_{(\alpha,a,t)}(u) du \tag{39}$$

Since $\pi_\sigma(\alpha, a, t)$ is positive decreasing in $\sigma$, $\phi_{(\alpha,a,t)}(Y)$ is positive and decreasing and $\Phi_{(\alpha,a,t)}(Y)$ is an increasing, concave, differentiable function of $Y$. Consider the set $\hat{A}_k$ of optimal investment choices for firm $k$ ($k \in K$)

$$\hat{A}_k(\alpha, a, t) = \text{argmax}\left\{ \sum_{s\in S} p_s(\zeta_k, a^{-k})\Phi_{(\alpha,a,t)}(Y_s) - \zeta_k \;\middle|\; 0 \le \zeta_k \le \sum_{i\in I} w_0^i - \epsilon \right\}$$

when the investments of other firms are $a^{-k}$ and $\Phi_{(\alpha,a,t)}$ is firm $k$'s utility function. By Assumption $\mathrm{C}_k$ and Lemma 2, $\hat{A}_k(\alpha, a, t)$ is a convex-valued, upper hemi-continuous correspondence. To ensure that the second component of $\Gamma$ lies in $D_\epsilon^K$, we project $\hat{A}(\alpha, a, t) = \prod_k \hat{A}_k(\alpha, a, t)$ onto $D_\epsilon^K$. The second component of the correspondence $\Gamma$ is thus

$$\tilde{a} \in \tilde{A}(\alpha, a, t) = \text{proj}_{D_\epsilon^K}(\hat{A}(\alpha, a, t))$$

and the third component is defined by

$$\tilde{t}_k(\alpha, a, t) = \min\{\theta_k \in \mathbb{R}_+ \,|\, P_{\mathbf{1}}(\alpha, a, t)y^k - a_k + \theta_k \ge 0\}, \quad k \in K$$

Since $P_{\mathbf{1}}(\alpha, a, t)\, y^k > 0$, $\tilde{t}_k(\alpha, a, t) \le a_k$ and since $a \in D_\epsilon^K$, $\tilde{t}(\alpha, a, t) = (\tilde{t}_k(\alpha, a, t))_{k\in K}$ lies in $D_\epsilon^K$.

By the Kakutani fixed point theorem, $\Gamma$ has a fixed point $(\alpha^*, a^*, t^*)$. Let us show that for $\epsilon$ sufficiently small, $(a^*, x^*, \pi^*, \tau^*, t^*)$, where $x^* = x(\alpha^*, a^*, t^*)$, $\tau^* = \tau(\alpha^*, a^*, t^*)$, and $\pi^* = \pi(\alpha^*, a^*, t^*)$, is an FEUT equilibrium in which the common utility function of the firms is $\Phi_{(\alpha^*,a^*,t^*)}$, for all $k \in K$.

By construction properties (iii) and (iv) of an FEUT equilibrium are satisfied, so that it remains to show that (i) and (ii) hold. To show that (i) is satisfied, we show that $\xi_i(\alpha^*, a^*, t^*) = 0$ for all $i$. This, combined with the first-order conditions for the maximum problem (38), implies that for all $i$, $x^{i*}$ is the solution of agent $i$'s maximum problem (i).

Let $\xi_i^* = \xi_i(\alpha^*, a^*, t^*)$, $i \in I$, and suppose that for some $i$, $\xi_i^* > 0$. Since $\sum_{i\in I} \xi_i^* = 0$, there must exist an agent $j$ for whom $\xi_j^* < 0$. Since

$$\alpha_j^* \left(1 + \sum_{i\in I} \max\{\xi_i^*, 0\}\right) = \alpha_j^* \tag{40}$$

by the fixed-point property and since $(1 + \sum_{i\in I} \max\{\xi_i^*, 0\}) > 1$, (40) implies that $\alpha_j^* = 0$. By monotonicity of preferences this implies that $x^{j*} = 0$. Since $\tau^* < 1$, $w_0^j > 0$ and the income from the firms are non-negative, the excess expenditure $\xi_j^* > 0$, which is a contradiction. Thus $\xi_i^* \le 0$ for all $i$ and since $\sum_{i\in I} \xi_i^* = 0$, $\xi_i^* = 0$ for all $i \in I$.

It remains to show that (ii) holds for the estimated social utility function $\Phi_{(\alpha^*,a^*,t^*)}$. The only way this might not be satisfied is if $a^*$ is the projection of some $\hat{a}$ in $\hat{A}(\alpha^*,a^*,t^*)$ onto $D_\epsilon^k$ which does not coincide with $\hat{a}$. This would imply that $\sum_{k\in K} a_k^* = \sum_{i\in I} w_0^i - \epsilon$: let us show that for $\epsilon$ sufficiently small this is impossible. If $\sum_{k\in K} a_k^* = \sum_{i\in I} w_0^i - \epsilon$, since all marginal rates of substitution are equalized, date 1 consumption increases with aggregate output for all agents, no agent consumes more than $\epsilon$ at date 0 and one agent consumes at least $\underline{Y}/I$ in the worst outcome at date 1, the vector of discount factors associated with the fixed point of the correspondence $\Gamma$ with domain $\mathbb{D}_\epsilon$ is bounded by

$$\pi_s^* \leq b_\epsilon \equiv \max_{i\in I} \frac{u_1^{i\prime}(\underline{Y}/I)}{u_0^{i\prime}(\epsilon)}, \quad s \in S$$

This implies that $\phi_{(\alpha^*,a^*,t^*)} \leq b_\epsilon$ and $\Phi_{(\alpha^*,a^*,t^*)} \leq b_\epsilon(Y - \underline{Y})$, $\forall\ Y \in [\underline{Y}, \overline{Y}]$. If an optimal choice $\hat{a}_k$ in $\hat{A}(\alpha^*,a^*,t^*)$ is such that $\hat{a}_k > b_\epsilon(\overline{Y} - \underline{Y})$ then $\sum_{s\in S} p_s(\hat{a}_k, a^{*-k})\Phi_{(\alpha^*,a^*,t^*)} - \hat{a}_k < 0$, while the objective of firm $k$ is positive if $a_k = 0$. Thus the optimal choice is such that $\hat{a}_k \leq b_\epsilon(\overline{Y} - \underline{Y})$. Since, when $\epsilon \to 0$, $b_\epsilon \to 0$, for $\epsilon$ sufficiently small, $\hat{a} = (\hat{a}_k)_{k\in K} \in \mathrm{Int}\Delta_\epsilon^K$ and coincides with its projection $a^*$ on $D_\epsilon^K$, contradicting the assumption that $a^*$ is on the boundary of $D_\epsilon^K$. Thus for all $k \in K$, $a_k^*$ is the solution of the maximum problem in (ii) when firm $k$'s estimated social utility is $\Phi_{(\alpha^*,a^*,t^*)}$, and thus $(a^*, x^*, P^*, \tau^*, t^*)$ is an FEUT equilibrium.

Note that in the case where the utility functions $(U^i)_{i\in I}$ are known the proof is the same except that the interpolated function $\Phi$ in (39) is replaced by the known sup-convolution function. □

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