DISTRIBUTING THE BENEFITS FROM THE COMMONS: A SQUARE-ROOT FORMULA

Joaquim Silvestre

Working Paper Series #95-06

Department of Economics
University of California
Davis, California 95616-8578

DISTRIBUTING THE BENEFITS FROM THE COMMONS: A SQUARE-ROOT FORMULA

Joaquim Silvestre

Working Paper Series No. 95-06
February 1995

ABSTRACT

"Distributing the Benefits from the Commons: A Square-Root Formula,"

by

Joaquim Silvestre

University of California, Davis

How should the benefits of the commons, say a publicly owned fishing resource, be distributed? A first possibility is equal division among the population. A second option is to distribute them among the people who actually exploit the resource in proportion to their activity level: this is the "land to the tiller" view. A third approach is the "usufruct" view, by which a consumer of the fruits of the commons ends up contributing the average cost, whithout generating incomes for nonconsumers.

The usufruct and "land to the tiller" views are polar opposites. One could consider intermediate positions where a fraction σ of the benefits is distributed among consumers in proportion to their consumption, and the fraction $1-\sigma$ is distributed among fishers in proportion to their fishing effort.

The paper singles out a particular value for σ based on equalizing the "rate of return," defined as follows. Consumers are the direct users of the fruits of the resource: they contribute numeraire (transferred to the fishers) and obtain fish in return. A fisher contributes time and obtains numeraire in return. It turns out that, if the "return ratios"

$$\frac{\text{VALUE OF RETURN}}{\text{VALUE OF CONTRIBUTION}}$$

are equalized across persons, fishers and consumers alike, then a particular value of σ results, namely:

$$\sigma^* = \frac{\sqrt{\text{VALUE OF TOTAL OUTPUT OF FISH}}}{\sqrt{\text{VALUE OF TOTAL OUTPUT OF FISH}} + \sqrt{\text{VALUE OF TOTAL FISHERS' LABOR}}} \quad .$$

"Distributing the Benefits from the Commons: A Square-Root Formula,"

by

Joaquim Silvestre*

University of California, Davis

January, 1995, version

1. Introduction

How should the benefits of the commons be distributed? I approach this normative question from the viewpoint of ownership, rather than income redistribution. In other words, I do not contemplate using the benefits from the commons for equalizing incomes or for helping disadvantaged groups. Think of middle-class fishers supplying food to middle-class households. The fishery is used as a metaphor for the standard common-pool situation: the analysis can be extended to a large class of multilateral production externalities.[1] The resource is understood to be socially owned. No historical rights exist and inefficiency is not an issue: visualize, for instance, a new fishing operation that is going to be efficiently carried.

The benefits from the commons are defined as the valuation of the natural resource at an efficient allocation (Sections 2.3 and 2.4). Essentially the same valuation appears in different scenarios. The valuation is only implicit in the

---

* The author is indebted to Andreu Mas-Colell and John E. Roemer. He has also benefited from conversations with Marc Fleurbaey. The usual *caveat* applies in full force. Financial support from the *Cátedra Gumersindo de Azcárate*, funded by the *Ministerio de Asuntos Sociales* in the *Universidad Carlos III de Madrid* , is gratefully acknowledged.

[1] The model is static, following John Chipman (1970). Static models of fisheries can be found in Martin Weitzman (1974) and Partha Dasgupta and Geoffrey Heal (1979). See Silvestre (1993) for a comparison with the more usual dynamic analysis of fisheries described, e.g., in Colin Clark (1976, 1990).

case of spontaneous cooperation among a well-defined group of fishers, or in the case of a fishery regulated by nontransferable fishing quotas, but it is explicit in other instances.

First, if the fishery is operated as a unitized, compètitive firm, then its profits express the valuation of the resource. This magnitude would also coincide with the revenue obtained by auctioning off the resource. Second, let the fishery be regulated by means of marketable permits. Then the valuation coincides with the amount of permits issued multiplied by the equilibrium price in the resale ("secondary") market. Finally, imagine that the fishing activity is subjected to linear Pigovian taxes. Then the valuation equals the total receipts of the tax bureau.

The benefits from the commons can be distributed in various ways. A first possibility is to divide them equally among the population: this option, under the name of equal benefit solution, is discussed to some extent in Roemer and Silvestre (1987) and Silvestre (1994). As an example, consider the distribution of oil rents by the State of Alaska to its residents. The basic idea is that, if there are one million people in society, then every person gets, at the end of the year, a check for one millionth of the benefits. Alternatively, an efficiency-inducing amount of transferable fishing permits would be issued and distributed uniformly among the members of society, who would then trade them in the secondary market. Another scheme in the same spirit would assign all fishing profits to the public treasury and, eventually, to the provision of public goods.

The equal-benefit solution presupposes that an a priori well defined group of people are the members of society. But there is some degree of artificiality in assigning the ownership of natural resources to a particular nation-state rather than to humankind.

The equal benefit solution displays another unattractive feature, namely that it gives the same rights to users and nonusers of the resource, and, thus, it transfers income from the former to the latter. An example will illustrate. Suppose that it takes no effort to catch fish. Let there be two people in society: person one, who enjoys fish, and, thus, spends some time fishing, and person two, who does not. According to the equal benefit solution, person one must pay person two, because of two's property rights on the fish population, despite the fact that person two is in no way involved in the capture or consumption of fish. Person two plays the role of a shareholder of a privately owned corporation, rather than that of the coowner of a publicly owned resource. For this reason, the equal benefit solution reflects "equal private ownership" rather than public ownership. Genuine public ownership of the resource should mean that person one, who has a use for it, freely enjoys the resource without having to compensate person two.

A second option is to adopt a "land to the tiller" view and identify public ownership with the granting of property rights to the people who actually exploit the resource. This may be justifiable for several reasons, including the possible benefits of decentralization and the concentration of ownership, as well as potential improvements in the distribution of income. But the position is hard to defend if one leaves these issues aside: in fact, it assigns all property rights to a small group, in conflict with the idea that the resource belongs to everybody. Think, in the extreme case, of a large producer who exploits a publicly owned resource, say, mineral deposits or timber in federal lands.

This leads to a "usufruct" view of resource ownership, which, in its purest form, implies that a consumer of the fruits of the publicly owned resource should end up paying exactly the average cost of production, whithout generating incomes for nonconsumers. Because the market price reflects the marginal cost of the product, higher than the average cost, this requires that the benefits of the fishery be distributed to consumers. The distribution must then in proportion to consumption, so that if a person consumes twice as much fish as another one, then the first one should receive a profit share twice as large. This idea leads to the proportional solution discussed in Roemer and Silvestre (1987, 1993).

In some sense, the proportional solution is the polar opposite to the "land to the tiller" view mentioned above. All benefits accrue to fishers in the latter, and to consumers in the former. One could consider intermediate positions where a fraction σ of the benefits is distributed among consumers in proportion to their consumption, and the rest (i.e., the fraction $1-\sigma$) is distributed among fishers in proportion to their fishing effort. The "land to the tiller" approach, then, corresponds to the value $\sigma = 0$, whereas the proportional solution sets $\sigma = 1$. A one-parameter family of solutions is obtained by letting σ range over the unit interval.

Section 3 below singles out a particular value for σ based on equalizing the "rate of return" defined as follows. Consumers are the direct users of the fruits of the resource, but instead of directly contributing a productive input, they transfer numeraire to the fishers. Suppose that the the "return ratios"

$$\frac{\text{VALUE OF RETURN}}{\text{VALUE OF CONTRIBUTION}}$$

are equalized across persons, with the understanding that a consumer contributes numeraire and obtains fish in return, while a fisher contributes time and obtains numeraire in return. In other words, the value of the good that one person (consumer or fisher) obtains is proportional to the value of the good that a person contributes. This means, in particular, that the ratio of the income obtained to the value of the input contributed by a fisher is equal, not only to the corresponding ratio of another fisher, but also to the value of a consumer's consumption of fish per unit of numeraire spent. The return ratios are equalized within groups (consumers or fishers) at any value of σ: the equal-rate-of-return condition extends this idea across the two groups.

It turns out (Theorem below) that the condition of equal rate of return induces a particular value for the parameter σ, namely:

$$\sigma^* = \frac{\sqrt{\text{VALUE OF TOTAL OUTPUT OF FISH}}}{\sqrt{\text{VALUE OF TOTAL OUTPUT OF FISH}} + \sqrt{\text{VALUE OF TOTAL FISHERS' LABOR}}} .$$

This gives consumers over one half of the benefits from the commons, reflecting the view that the social valuation of the product of the resource is higher than that of the input applied to it.

2. The model

2.1. Agents, goods, resources, technology.

There are F+C economic agents: F of them are fishers, indexed 1,...,F; C of them are consumers, indexed F+1,..., F+C.

There are three goods. The first good is gold, measured in ounces: gold is, in what follows, the numeraire good. The second good is leisure time (or labor time), measured in hours; its individual final use (resp. supply) is denoted x_i (resp. L_i), and its aggregate supply is denoted L, i.e., $L = \sum_{i=1}^{F+C} L_i$. The third good is fish, measured in pounds; consumer i's individual consumption of fish is denoted y_i, and the aggregate quantity supplied and consumed is denoted y, i.e., $y = \sum_{i=F+1}^{F+C} y_i$. Agent i's individual final holdings of gold are denoted m_i ($i = 1,...,F+C$).

Labor time and gold are initially available in the amounts T and ω, respectively. They are privately owned, but consumers do not own labor time. Denote by T_i ($i = 1,...,F$) fisher i's labor endowment, and by ω_i ($i = 1,...,F+C$)

agent i's gold endowment. Then $T = \Sigma_{i=1}^{F} T_i$ and $\omega = \Sigma_{i=1}^{F+C} \omega_i$. Fisher i's initial endowment of time T_i, amount of labor supplied L_i and leisure time enjoyed x_i are related by the equality $L_i + x_i = T_i$.

The preferences of fisher i (i = 1,...,F) are represented by a utility function $u_i(m_i, x_i)$, with quantities of gold and leisure time as arguments. The preferences of consumer i (i = F+1,...,F+C) are represented by a utility function $u_i(m_i, y_i)$, with quantities of gold and fish as arguments. The utility functions are assumed to be differentiable, strictly quasiconcave and strictly monotonic in the interior of the relevant orthant.

Society's technological possibilities are described by a production function $f(L)$, differentiable, strictly concave and satisfying $f(0) = 0$.

2.2. Efficiency

An allocation is a 2(F+C)-dimensional vector

$$(m_1, x_1, \ldots, m_F, x_F, m_{F+1}, y_{F+1}, \ldots, m_{F+C}, y_{F+C}).$$

An allocation is <u>feasible</u> if: $\Sigma_{i=1}^{F+C} m_i = \omega$ and $y = f(L)$, where $y = \Sigma_{i=F+1}^{F+C} y_i = f(L)$, and $L = \Sigma_{i=1}^{F} (T_i - x_i)$. An allocation is <u>interior</u> if $x_i > 0$ (i = 1,...,F), $y_i > 0$ (i = F+1,...,F+C) and $m_i > 0$ (i = 1,..., F+ C), i.e., the consumption vector of each agents has the two relevant components positive. At an efficient, interior allocation one must have that, for any pair i, h, where i is a fisher and h is a consumer,

$$\frac{\frac{\partial u_i}{\partial x_i}}{\frac{\partial u_i}{\partial m_i}} = \frac{\frac{\partial u_h}{\partial y_h}}{\frac{\partial u_h}{\partial m_h}} f'(L), \quad i = 1,\ldots,F,\ h = F+1,\ldots,F+C.$$

At an efficient, interior allocation, a valuation of the goods is given by the vector of support prices (1,w, p) defined by:

$$p = \frac{\frac{\partial u_h}{\partial y_h}}{\frac{\partial u_h}{\partial m_h}}, \text{ for any } h \text{ in } \{F+1,\dots,F+C\},$$

and:

$$w = \frac{\frac{\partial u_i}{\partial x_i}}{\frac{\partial u_i}{\partial m_i}}, \text{ for any } i \text{ in } \{1,\dots,F\}.$$

Support prices allow for profit maximization and constrained utility maximization. The valuation of the resource provided by the support prices is $py - wL$.

2.3. Property relations.

As just mentioned, the endowments of labor-time and gold are privately owned. The technology, on the contrary, is publicly owned.

The public ownership of the technology can be understood, à la McKenzie (1959), as the public ownership of a nonmarketed input which is implicit in the production function $f(L)$. Recall that the McKenzie construction associates to a strictly convex production function $f(L)$ a two-input, constant return to scale production function $F(L,\xi)$, where ξ is an implicit input, available in one unit, whose marginal value product equals the profits $pf(L) - wL$ obtained with the original production function $f(L)$ under competitive conditions.[2] McKenzie (1959)

---

[2] In detail, define the two-input production function as $F(L,\xi) = \xi\, f(\frac{L}{\xi})$, if $\xi >$ 0, and $F(L,\xi) = 0$, if $\xi = 0$. Clearly, the corresponding production set is a cone. One computes: $\frac{\partial F}{\partial \xi} = f(\frac{L}{\xi}) + \xi f'(\frac{L}{\xi})[-\frac{L}{\xi^2}]$, equal to $f(L) - Lf'(L)$ when $\xi = 1$.

interpreted the implicit input as a privately owned "entrepreneurial input." But, in the joint exploitation of a common pool resource, it is the limited availability of the resource itself that plays a major role in the decreasing returns to the labor input. If the role of entrepreneurial capacity is negligible relative to that of the jointly exploited resource, then one may identify the implicit input with the resource.

2.4. The unitized firm vs. individual exploitation

Fish is traded in a perfectly competitive fish market. As indicated in the Introduction, we may consider three different organizations of the fishery, each providing an interpretation of the benefits from the resource. First, a unitized, collectively owned firm operates the fishery, buying fishers' labor in a competitive labor market. The market wage is denoted by w. As just noted, the resulting competitive profits pf(L) - wL can be interpreted, à la McKenzie, as the competitive valuation of the resource. A particular distribution of the shares in the firm's profits implements a distribution of the value of the resource.

But suppose that each fisher operates alone and sells her catch in the fish market. Assume that the amount of fish caught by fisher i is proportional to the amount L_i of time that she spends fishing, i.e., all fishers are equally skilled and equally lucky, so that i's output is $\frac{f(L)}{L} L_i$. The attainment of an efficient outcome is then often contingent to policy intervention. For instance, a central authority can issue an efficient number of fishing permits, which can then be

---

Moreover, $\frac{\partial F}{\partial L} = \xi\, f'(\frac{L}{\xi})\frac{1}{\xi}$, equal to $f'(L)$ when $\xi = 1$. By perfect competition, $f'(L) = \frac{w}{p}$. Thus, $\frac{\partial F}{\partial \xi} = f(L) - \frac{w}{p} L$, i.e., the profit is the marginal product of the implicit input.

traded in a secondary market. The value of the resource is now the market value of the total number of permits issued. An initial allocation of the marketable permits implements a distribution of the value of the resource.

An alternative policy tool is the imposition of a linear, Pigovian tax. The value of the resource is then measured by the aggregate tax receipts. A lumpsum distribution of the receipts implements a distribution of the value of the resource.

A particular distribution of the value of the resource is formalized here as a vector $\theta = (\theta_1,\ldots,\theta_F,\theta_{F+1},\ldots,\theta_{F+C})$ such that $\theta_i \geq 0$ and $\Sigma_{i=1}^{F+C}\theta_i = 1$. The language and formal definitions reflect the interpretation of a unitized, competitive firm. The reader is invited to translate, as an exercise, the definitions and results into the "marketable permits" and "Pigovian tax" interpretations.

3. Public ownership solutions

3.1 Preliminary definition

As just noted, I formalize the allocation that corresponds to a particular distribution vector θ as a Walrasian equilibrium with profit shares defined by θ. Formally:

Definition: Let the vector $(\theta_1,\ldots,\theta_{F+C})$, where $\theta_i \geq 0$, $i = 1,\ldots,M$, and $\Sigma_{i=1}^{M}\theta_i = 1$, be given. A vector $(\overline{m}_1,\overline{L}_1,\ldots,\overline{m}_F,\overline{L}_F,\overline{m}_{F+1},\overline{y}_{F+1},\ldots,\overline{m}_{F+C},\overline{y}_{F+C},w,p)$ is a Walrasian equilibrium for the benefit distribution $(\theta_1,\ldots,\theta_M)$ if, writing $\overline{L} = \Sigma_{i=1}^{F}\overline{L}_i$, $y = \Sigma_{i=F+1}^{F+C}\overline{y}_i$, and $\Pi = p\overline{y} - w\overline{L}$,

(i) for $i = 1,\ldots, F$, $(\overline{m}_i, T_i - \overline{L}_i)$ maximizes $u_i(m_i, x_i)$ subject to

$$wx_i + m_i = \omega_i + wT_i + \theta_i\Pi;$$

(ii) for $i = F+1,\ldots,F+C$, $(\overline{m}_i, \overline{y}_i)$ maximizes $u_i(m_i, y_i)$ subject to

$$p y_i + m_i = \omega_i + \theta_i \Pi \ ;$$

(iii) $\overline{y} = f(\overline{L})$;

(iv) $pf(\overline{L}) - w\overline{L} \geq pf(L) - wL$, for all $L \geq 0$;

(v) $\Sigma_{i=1}^{F+C} \overline{m}_i = \omega$.

It is clear, from the first fundamental theorem of welfare economics, that such an equilibrium is efficient, and that the allocation is supported by the price vector (1,w, p).

3.2. The equal benefit solution

As described in the Introduction, the equal benefit solution divides the benefits equally among the members of society. Formally:

Definition: A vector $(m_1, L_1, \ldots, m_F, L_F, m_{F+1}, y_{F+1}, \ldots, m_{F+C}, y_{F+C}, w, p)$ is an equal-benefit solution if it is Walrasian equilibrium for the benefit distribution $(\theta_1, \ldots, \theta_F, \theta_{F+1}, \ldots, \theta_{F+C}) = (\frac{1}{F+C}, \ldots, \frac{1}{F+C}, \frac{1}{F+C}, \ldots, \frac{1}{F+C}, \frac{1}{F+C})$.

3.3. A one-parameter family of public ownership solutions

Definition: Given $\sigma \in [0,1]$, a vector $(m_1, L_1, \ldots, m_F, L_F, m_{F+1}, y_{F+1}, \ldots, m_{F+C}, y_{F+C}, w, p)$ is a σ-public-ownership solution if it is an Walrasian equilibrium for the benefit distribution

$$(\theta_1, \ldots, \theta_F, \theta_{F+1}, \ldots, \theta_{F+C}) = ((1-\sigma)\frac{L_1}{L}, \ldots, (1-\sigma)\frac{L_F}{L}, \sigma \frac{y_{F+1}}{y}, \ldots, \sigma \frac{y_{F+C}}{y}).$$

If $\sigma = 0$, then $(\theta_1, \ldots, \theta_F, \theta_{F+1}, \ldots, \theta_{F+C}) = (\frac{L_1}{L}, \ldots, \frac{L_F}{L}, 0, \ldots, 0)$, i.e., all the benefits from the commons accrue to the fishers, and a fisher's benefit share equals her labor share: this captures the "land to the tiller" approach mentioned in the Introduction. If, on the contrary, $\sigma = 1$, then $(\theta_1, \ldots, \theta_F, \theta_{F+1}, \ldots, \theta_{F+C}) = (0, \ldots, 0, \frac{y_{F+1}}{y}, \ldots, \frac{y_{F+C}}{y})$, i.e., all the benefits from the commons accrue to consumers, and a consumer's benefit share equals her consumption share: this is the "proportional solution."

3.4. The equalization of the rate of return

As mentioned in the Introduction, fishers derive a return in gold (numeraire) from a contribution of labor time, whereas consumers derive a return equal to the value of fish they consume from a contribution in gold. Evaluated at the support prices, the magnitudes are:

fisher i's return (gold): $m_i - \omega_i$;

fisher i's contribution (labor time): wL_i;

thus, fisher i's rate of return is: $\dfrac{m_i - \omega_i}{wL_i}$;

consumer h's return (fish): py_h;

consumer h's contribution (gold): $\omega_h - m_h$;

thus, consumer h's rate of return is: $\dfrac{py_h}{\omega_h - m_h}$.

The equal-rate-of-return condition requires the equalization of the rates of return, or, synonymously, it requires that all returns be proportional to contributions. Formally:

Definition: An efficient, interior allocation $(x_1, m_1, \ldots, x_F, m_F, y_{F+1}, m_{F+1}, \ldots, y_{F+C}, m_{F+C})$ is an equal-rate-of-return solution if for each pair (i,h), where i is a fisher and h is a consumer,

$$\frac{m_i - \omega_i}{wL_i} = \frac{py_h}{\omega_h - m_h}, \tag{3.1}$$

where $L_i = T_i - x_i$, $i = 1, \ldots, F$, and where the price vector (1, w, p) supports the allocation.[3]

---

[3] For simplicity, the definition is restricted to interior allocations. The analysis can easily be extended to cover boundary points. Define an equal-rate-of-return solution by the conditions: (a) $y_h > 0$ iff $\omega_h - m_h > 0$; (b) $L_i > 0$ iff $m_i - \omega_i$

The main result of this paper is that the equal-rate-of return condition implies that consumers, in the aggregate, get a fraction σ^* of the benefits of the commons (and fishers a fraction $1 - \sigma^*$), where $\sigma^* = \frac{\sqrt{py}}{\sqrt{py}+\sqrt{wL}}$. Within each group, benefit shares are proportional to labor shares (for fishers) or consumption shares (for consumers.) Formally:

Theorem: The efficient, interior allocation $(m_1,x_1,\dots,m_F,x_F,m_{F+1},y_{F+1},\dots,m_{F+C},y_{F+C})$, with support prices $(1, w, p)$, and with aggregate quantities $y = \Sigma_{i=F+1}^{F+C} y_i$ and $L = \Sigma_{i=1}^{F} L_i$, where $L_i = T_i - x_i$, $i = 1,\dots,F$, is an equal-rate-of-return solution if and only if $(m_1,L_1,\dots,m_F,L_F,m_{F+1},y_{F+1},\dots,m_{F+C},y_{F+C},w, p)$ is a σ^*-public-ownership equilibrium, for

$$\sigma^* = \frac{\sqrt{py}}{\sqrt{py}+\sqrt{wL}} \quad .$$

Proof. The result will follow after showing that condition (3.1) obtains if and only if:

$$m_i = \omega_i + wL_i + (1-\sigma^*)\frac{L_i}{L}(py - wL),\ i = 1,\dots,F, \tag{3.2}$$

and:

$$m_h = \omega_h - py_h + \sigma^*\frac{y_h}{y}(py - wL),\ h = F+1,\dots,F+C. \tag{3.3}$$

First I prove that (3.1) implies (3.2). Write $B = \Sigma_{i=1}^{F} (m_i - \omega_i)$, the total amount of gold transferred from consumers to fishers. Note that (3.1) implies that:

$$\frac{m_i - \omega_i}{wL_i} = \frac{B}{wL},\quad i = 1,\dots F, \tag{3.4}$$

and:

> 0; and (c) if $L_i > 0$ and $\omega_h - m_h > 0$, then (3.1) holds. The Theorem below then holds with minor and obvious language changes.

$$\frac{py_h}{\omega_h - m_h} = \frac{py}{\omega - \Sigma_{k=F+1}^{F+C} m_k} \quad , \quad h = F+1,\ldots,F+C. \tag{3.5}$$

Because $\Sigma_{k=1}^{F+C} m_k = \omega$ at an efficient allocation, we have that $\omega - \Sigma_{k=F+1}^{F+C} m_k = B$.

Thus, (3.5) can be written:

$$\frac{py_h}{\omega_h - m_h} = \frac{py}{B} \quad , \quad h = F+1,\ldots,F+C. \tag{3.6}$$

From (3.1), (3.4) and (3.6), we obtain: $B^2 = pywL$, i.e., $B = \sqrt{pywL}$. Thus, (3.4) and (3.6) become:

$$\frac{m_i - \omega_i}{wL_i} = \frac{\sqrt{py}}{\sqrt{wL}}, \quad i = 1,\ldots F, \tag{3.7}$$

$$\frac{py_h}{\omega_h - m_h} = \frac{\sqrt{py}}{\sqrt{wL}} \quad , \quad h = F+1,\ldots,F+C. \tag{3.8}$$

Consider fisher i, i = 1,...,F. From (3.7),

$$\begin{aligned} m_i - \omega_i &= wL_i \frac{\sqrt{py}}{\sqrt{wL}} \\ &= wL_i + wL_i\left(\frac{\sqrt{py}}{\sqrt{wL}} - 1\right) \\ &= wL_i + wL_i\left(\frac{\sqrt{py} - \sqrt{wL}}{\sqrt{wL}}\right) \\ &= wL_i + wL_i\left(\frac{\sqrt{py} - \sqrt{wL}}{\sqrt{wL}}\right) \frac{\sqrt{py} + \sqrt{wL}}{\sqrt{py} + \sqrt{wL}} \\ &= wL_i + \frac{wL_i}{wL} \frac{wL}{\sqrt{wL}} \frac{py - wL}{\sqrt{py} + \sqrt{wL}} \\ &= wL_i + \frac{L_i}{L}(1-\sigma^*)(py - wL), \end{aligned}$$

proving that (3.2) holds. In a parallel manner, consider consumer h, h = F+1, ...,F+C. From (3.8) we have that:

$$py_h \frac{\sqrt{wL}}{\sqrt{py}} = \omega_h - m_h \quad ,$$

i.e.,

$$\begin{aligned} m_h &= \omega_h - py_h + py_h\left(1 - \frac{\sqrt{wL}}{\sqrt{py}}\right) \\ &= \omega_h - py_h + py_h \frac{\sqrt{py} - \sqrt{wL}}{\sqrt{py}} \end{aligned}$$

$$= \omega_h - py_h + py_h \frac{\sqrt{py} - \sqrt{wL}}{\sqrt{py}} \frac{\sqrt{py} + \sqrt{wL}}{\sqrt{py} + \sqrt{wL}}$$

$$= \omega_h - py_h + \frac{py_h}{py} \frac{py}{\sqrt{py}} \frac{py - wL}{\sqrt{py} + \sqrt{wL}}$$

$$= \omega_h - py_h + \frac{y_h}{y} \sigma^* (py - wL),$$

showing that (3.3) is satisfied. The support property of prices guarantees utility maximization and profit maximization. Thus, writing $\theta_i = \frac{L_i}{L}(1-\sigma^*)$, $i = 1,\ldots,F$, and $\theta_h = \frac{y_h}{y}\sigma^*$, $h = F+1,\ldots,F+C$, we conclude that

$(m_1, L_1, \ldots, m_F, L_F, m_{F+1}, y_{F+1}, \ldots, m_{F+C}, y_{F+C}, w, p)$ is a σ^*-public-ownership equilibrium for $\sigma^* = \frac{\sqrt{py}}{\sqrt{py} + \sqrt{wL}}$.

Conversely, consider a σ^*-public ownership equilibrium for the stated σ^*. From (3.2), we have that

$$m_i - \omega_i = wL_i \left(1 + (1-\sigma^*)\frac{py - wL}{wL}\right)$$

$$= wL_i \left(1 + \frac{\sqrt{wL}}{\sqrt{py} + \sqrt{wL}} \frac{(\sqrt{py} + \sqrt{wL})(\sqrt{py} - \sqrt{wL})}{wL}\right)$$

$$= wL_i \left(1 + \frac{\sqrt{py} - \sqrt{wL}}{\sqrt{wL}}\right)$$

$$= wL_i \frac{\sqrt{py}}{\sqrt{wL}},$$

i.e.,

$$\frac{m_i - \omega_i}{wL_i} = \frac{\sqrt{py}}{\sqrt{wL}}, \text{ for } i = 1,\ldots,F.$$

In a parallel manner, from (3.3) we obtain:

$$\frac{py_h}{\omega_h - m_h} = \frac{\sqrt{py}}{\sqrt{wL}} \quad , \quad h = F+1,\ldots,F+C.$$

Thus, the equal-rate-of-return equalities (3.1) are satisfied. Efficiency follows from the first fundamental theorem, and the supporting property of prices from utility maximization and profit maximization. Q.E.D.

4. Concluding remarks

Common pool resources often involve two distinct groups of economic agents: the producers and the consumers. Many real-life instances of benefit distribution in common-pool resources resemble the "land to the tiller" solution, where producers are the main beneficiaries, sometimes combined with the "equal benefit solution," perhaps in the form of royalties that accrue to the treasury, (although some situations where users pay average cost could be interpreted in terms of the proportional solution). This paper advocates distributing a fraction of the benefits to the final users of the resource. After all, the social surplus derived from the resource is created by both on the productive ability of the exploiters and the presence of consumers who can use its fruits.

A simple principle of equalizing the rate of return, both within each group and across the two groups, yields a surprising formula for the intergroup distribution. The formula, involving the square roots of the value of output and the value of input, only requires neoclassical assumptions on utility and production functions, and does not depend on particular functional forms.

Of course, this is not the first time that a formula involving square roots has appeared in economics. The honor belongs no doubt to Johann Heinrich von Thünen's formula, engraved on his tombstone, for the "natural wage," see Joseph A. Schumpeter (1954, p 467).

REFERENCES

Chipman, John S. (1970) "External Economies of Scale and Competitive Equilibrium," Quarterly Journal of Economics 84 (3), 347-385.

Clark, Colin W. (1976, 1990) Mathematical Bioeconomics: The Optimal Management of Renewable Resources, (first edition: 1976, second edition: 1990) New York: John Wiley.

Dasgupta, Partha S. and Geoffrey M. Heal (1979) Economic Theory and Exhaustible Resources, Cambridge: Cambridge University Press.

McKenzie, Lionel W. (1959) "On the Existence of a General Equilibrium for a Competitive Economy," Econometrica 27, 54-71.

Roemer, John E., and Joaquim Silvestre (1987) "Public Ownership: Three Proposals for Resource Allocation," Department of Economics, University of California, Davis, Working Paper Series No. 307.

Roemer, John E., and Joaquim Silvestre (1993) "The Proportional Solution for Economies with both Private and Public Ownership," Journal of Economic Theory, 59 (2), 426-444.

Schumpeter, Joseph A. (1954) History of Economic Analysis, Oxford: Oxford University Press.

Silvestre, Joaquim (1993) "The Commons: Static and Dynamic Externalities," unpublished.

Silvestre, Joaquim (1994) "Economic Analysis of Public Ownership," Investigaciones Económicas 16(1), 19-66

Weitzman, Martin (1974) "Free Access vs. Private Ownership as Alternative Systems for Managing Public Property," Journal of Economic Theory 8(2), 225-234.