

**EXTENDED PARTIAL ORDERS:
A Unifying Structure for Abstract Choice Theory**

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Abstract *The concept of a strict extended partial order (SEPO) has turned out to be very useful in explaining (resp. **rationalizing**) non-binary choice functions. The present paper provides a general account of the concept of extended binary relations, i.e. relations between subsets and elements of a given universal set of alternatives. In particular, we define the concept of a weak extended partial order (WEPO) and show **how** it can be used in order to represent **rankings** of opportunity sets that display a “**preference** for opportunities.” We also clarify the relationship between SEPOs and WEPOs, which involves a non-trivial condition, called “strict **properness**.” Several characterizations of strict (and weak) properness are provided based on which we argue for properness as an appropriate condition demarcating “choice based” preference.*

1 Introduction

The concept of an extended strict partial order (SEPO) has been introduced in Nehring [1996] in order to explain the structure of non-binary choice functions. The notion of an extended partial order serves as an appropriate generalization of the concept of a transitive, but possibly incomplete preference among alternatives. In particular, **building** on the work of Aizerman [1985], and Aizerman and Malishevski [1981], it has been shown in Nehring [1996] that a choice function is rationalizable by a SEPO if and only if it satisfies two fundamental rationality conditions, contraction consistency and the so-called "Aizerman condition," or, equivalently, if and only if it satisfies Plott's [1973] famous path-independence condition.¹ The present paper introduces the notion of a *weak* extended partial order (WEPO) by substituting a straightforward reflexivity condition for the (non-trivial) irreflexivity condition characterizing SEPOs. It is shown that WEPOs can be used to represent the qualitative structure of rankings of opportunity sets that display a "preference for opportunities" (in a sense to be defined). Moreover, in a choice functional context, the subclass of "strictly proper" WEPOs rationalize "acceptability" of alternatives in a natural way that complements the rationalization of inferiority in terms of SEPOs. A number of characterizations of strict (and "weak") properness are given, based on which we argue that properness demarcates *choice* based weak preference.

The paper is organized as follows. Section 2 contains the basic definitions. In particular, we identify appropriate conditions of reflexivity, irreflexivity and transitivity for extended relations. Section 3 briefly summarizes the relation between SEPOs and choice functions established in Nehring [1996]. In Section 4, we establish an isomorphism between WEPOs and rankings of sets that display a "preference for opportunities." As a corollary, we obtain a characterization of a WEPO as the intersection of all its weak order extensions. Section 5 is devoted to the interrelation of WEPOs and SEPOs introducing the condition of strict properness. We also briefly discuss a weakening of properness that corresponds to a condition of "Irrelevance of Inessential Elements" (IIE) introduced in Nehring and Puppe [1996], and translate the analysis there to characterize strict and weak properness in terms of restrictions on the intersection representation. Section 6 addresses the question of how the results generalize to the case of arbitrary non-finite domains. For the general intersection representation of WEPOs we give a complete answer using an appropriate "regularity" condition. An analogous result for proper WEPOs seems to be much more difficult, and we only provide a partial solution to that problem.

¹For recent work on this class, see Bandyopadhyay [1986], Johnson [1990], Malishevski [1994] and Sertel [1988a, b].

2 Basic Concepts

Let X be an arbitrary set of alternatives and denote by $P^0(X) := P(X) \setminus \{\emptyset\}$ the set of all non-empty subsets of X . Elements of $P^0(X)$ are interpreted as *choice situations* or *opportunity sets*. An *extended (preference) relation* Q on X is a binary relation between subsets and elements of X , i.e. $Q \subseteq P^0(X) \times X$. For $A \in P^0(X)$ and $x \in X$, the statement " AQx " is interpreted as "the choice situation, or opportunity set, A is preferred to the (degenerate) choice situation, or (degenerate) opportunity set, $x (= \{x\})$ "² Consider the following basic properties of extended relations.

Reflexivity (REF) For all x, A ,

$$x \in A \Rightarrow AQx$$

Irreflexivity (IRR) For all x, A ,

$$AQx \Rightarrow [A \setminus x \neq \emptyset \text{ and } (A \setminus x)Qx]$$

Monotonicity (MON) For all x, A, B ,

$$[A \subseteq B \text{ and } AQx] \Rightarrow BQx.$$

Property REF is a straightforward generalization of standard reflexivity. Property MON seems to be very natural in our context. It states that having more options can never be harmful and thus reflects the absence of "effort-of-decision costs." Property IRR is a suitable generalization of irreflexivity. A choice-based justification for IRR is as follows. If A is Q -superior to x , then the "non- x " elements in A must be superior, i.e. $(A \setminus x)Qx$.

The following condition generalizes the notion of transitivity for extended relations. Note that for $A = \emptyset$ and $B = z$ it reduces to ordinary transitivity for (non-extended) binary relations.

Transitivity (TRA) For all $x, y, A \cup y, B$,

$$[(A \cup y)Qx \text{ and } BQy] \Rightarrow (A \cup B)Qx.$$

An extended relation satisfying monotonicity, transitivity and either reflexivity or irreflexivity will be called an *extended partial order (EPO)*. A reflexive EPO is called a *weak* extended partial order (WEPO), an irreflexive EPO is called a *strict* extended partial order (SEPO). Throughout, WEPOs will be denoted by the symbol R and SEPOs by the symbol P .

The simplest examples of EPOs are those that are *binary*, i.e. those that satisfy the following property.

²Throughout, singletons are identified with elements, i.e. for all $x \in X$, $\{x\}$ is identified with x .

Binariness (BIN) For all x, A ,

$$AQx \Rightarrow \exists y \in A : yQx.$$

A binary **EPO** Q is completely determined by its (non-extended) *base relation* $Q_b := Q \cap (X \times X)$. Note that a binary and monotone extended relation is reflexive (irreflexive, transitive) if and only if its base relation is reflexive (irreflexive, transitive) in the usual sense. In accordance with standard terminology we may thus define a *weak order* on X as a binary **WEPO** that is complete on $X \times X$. Similarly, a *linear order* on X is a binary **SEPO** P that is weakly connected on X in the sense that for all $x, y \in X$, $x \neq y$ implies $(xPy \text{ or } yPx)$.³

3 SEPOs and Non-binary Choice

The prominent role of SEPOs in the context of rationalizing non-binary choice functions has been established in Nehring [1996].⁴ Consider a choice function $C : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ defined on the domain $\mathcal{F}(X)$ of all finite and non-empty subsets of X . As usual, the term "choice function" refers to the assumption that, for all $A \in \mathcal{F}(X)$, $C(A) \subseteq A$. Any such choice function may be "rationalized" by an extended relation P in terms of the following condition. For all x and all $A \in \mathcal{F}(X)$,

$$(p) \quad APx \Leftrightarrow x \notin C(A \cup \{x\}).$$

Note that, given an extended relation P and a choice function C that are related by means of condition p , P automatically satisfies IRR (on $\mathcal{F}(X)$).

Theorem 3.1 (Nehring [1996]) *A choice function $C : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ is rationalizable by a SEPO $P \subseteq \mathcal{F}(X) \times X$ in the sense of condition p if and only if C satisfies for all x, A, B*

$$(\alpha) \quad x \in A \setminus C(A) \text{ and } A \subseteq B \Rightarrow x \notin C(B),$$

and

$$(\eta) \quad C(B) \subseteq A \subseteq B \Rightarrow C(A) \subseteq C(B).$$

In the literature, η is sometimes referred to as the Aizerman-condition. It is well known that on a finite domain α and η are together necessary and sufficient for a

³A note on terminology: sometimes a reflexive and transitive binary relation is called a *preorder*, whereas a *partial order* is in addition required to be antisymmetric. In this paper, given reflexivity, the term "partial order" is used synonymously for "preorder." Wherever assumed, antisymmetry is explicitly mentioned.

⁴Andrey Malishevski has informed us that he independently arrived at a concept essentially identical to that of a SEPO in a forthcoming paper (which has not yet been accessible to the authors).

choice function to be rationalizable by a set $\mathcal{P}$ of linear orderings in the following sense. For all A ,

$$C(A) = \bigcup_{P \in \mathcal{P}} \max_P A, \quad (3.1)$$

where $\max_P A$ denotes the (unique) best element in A with respect to the linear ordering P . The following result is thus a corollary of Theorem 3.1. Let $\mathcal{C}$ denote the set of all linear orderings on X .

Corollary 3.1 *Let X be finite. An extended relation P on X is a SEPO if and only if it is the intersection of all its linear extensions, i.e.,*

$$P = \bigcap \{Q \in \mathcal{L} : Q \supseteq P\}$$

In particular, the set of linear extensions is non-empty.

4 WEPOs and "Preference for Opportunities"

In this section it is shown how the concept of a *weak* EPO naturally arises in the context of ranking sets of alternatives in terms of their "opportunity value." Specifically, it will be shown that there is an isomorphism between WEPOs and rankings of opportunity :sets, or choice situations, that display a "preference for opportunities" (in an appropriately specified sense). The assumption here is that choice is performed in two stages. First, an opportunity set is determined from which in later stage of choice one particular alternative is chosen as the final outcome. As the simplest example consider the *indirect utility* preference $\succeq^R$ derived from some weak order R on X , which is defined as follows. For all A, B ,

$$A \succeq^R B \Leftrightarrow \forall y \in B \exists x \in A : x R y.$$

In particular, for all A and each alternative x ,

$$A \cup x \succ^R A \Leftrightarrow \forall y \in A : x P y, \quad (4.1)$$

where P denotes the asymmetric part of R . Clearly, since R is complete on X , $\succeq^R$ is complete on $P^0(X)$. However, indirect utility preferences neglect important aspects of decision making that may cause *incompleteness* of preference. For instance, a decision maker may be *uncertain* about the preferences that are relevant at the moment of choice of the alternative. More generally, as in a multi-attribute decision problem a decision maker may attach importance to different "viewpoints" reflected by different **rankings** of the alternatives. Accordingly, we consider the following generalization of (4.1). Let $\mathbf{t}$ denote a partial order on $P^0(X)$, i.e. a reflexive and transitive (but not necessarily, complete) binary relation among sets of alternatives.

Definition 4.1 We will say that $\succeq$ displays a preference for *opportunities* if and only if there exists a set $\mathcal{R}$ of weak orders on X such that for all x, A ,

$$A \cup x \succ A \Leftrightarrow \exists R \in \mathcal{R} \forall y \in A : xPy. \quad (4.2)$$

By (4.2), the addition of an alternative x to A is of positive value if and only if x is superior to A (in terms of indirect utility) for some "relevant" ordering $R \in \mathcal{R}$. Definition 4.1 thus captures the intuitive argument for the value of having choice put forward in Jones and Sugden [1982]. A particular interpretation of the set $\mathcal{R}$ of weak orders is as a decision maker's possible future preferences. In this case, condition (4.2) may be thought of expressing a "preference for flexibility" (see Kreps [1979]). As noted above, a more general interpretation of the set $\mathcal{R}$ is as different relevant "viewpoints" from which a decision maker evaluates alternatives. In this case, the term "preference for opportunities" seems to be appropriate.

Note that, given a set $\mathcal{R}$ of weak orders on X , (4.2) in effect imposes consistency conditions on the following extended *relation* on X derived from the ranking $\succeq$. Say that x is essential at A , denoted by AEx , if and only if $A \cup x \neq A$. The question to be studied in the following is: Which extended relations E are multi-preference rationalizable in the sense of Definition 4.1? In other words, what are the conditions on an extended relation E that imply the existence of a set $\mathcal{R}$ of weak orders on X such that

$$AEx \Leftrightarrow \exists R \in \mathcal{R} \forall y \in A : xPy$$

Obviously, the following two conditions are necessary for multi-preference rationalizability. For all x, A, B ,

$$AEx \Rightarrow x \notin A, \quad (4.3)$$

$$[A \subseteq B \text{ and } BEx] \Rightarrow AEx. \quad (4.4)$$

Condition (4.3) is straightforward. Condition (4.4) states that if x is essential at B , then x must be essential at any subset A of B (in Nehring and Puppe [1996], condition (4.4) is referred to as contraction consistency). Conditions (4.3) and (4.4) can be made more transparent by considering the corresponding "inessentiality" relation R_E which is defined as follows. For all x, A ,

$$AR_E x \Leftrightarrow \text{not } [AEx]$$

The following fact is easily established. It shows, in particular, that the "consistency" condition (4.4) exactly corresponds to the canonical property MON.

Fact 4.1 The extended relation E satisfies (4.3) if and only if R_E satisfies REF. Furthermore, E satisfies (4.4) if and only if R_E satisfies MON.

It is clear that E is multi-preference rationalizable in the sense of Definition 4.1 if and only if R_E is multi-preference rationalizable in the following sense: There exists a set of weak orders $\mathcal{R}$ on X such that for all x, A ,

$$AR_E x \Leftrightarrow \forall R \in \mathcal{R} : ARx. \quad (4.5)$$

The following theorem establishes that the class of extended relations that are multi-preference rationalizable is precisely the class of all WEPOs.

Theorem 4.1 *Let X be a finite set, and let R_E be an extended relation on X . R_E is multi-preference rationalizable (in the sense of (4.5)) if and only if R_E satisfies REF, MON and TRA, i.e. if and only if R_E is a WEPO.*

The proof of Theorem 4.1 is given in two steps. First, it is shown that a reflexive and monotonic extended relation R_E can be extended to a partial order on $P^0(X)$, weakly monotonic with respect to set inclusion, if and only if R_E satisfies TRA.⁵ In a second step, it is verified that the smallest extension $\succeq_E^*$ of R_E to $P^0(X)$ satisfies the necessary and sufficient conditions in order to apply a fundamental representation theorem due to Kreps [1979] which gives the desired set $\mathcal{R}$ of weak orders.

Lemma 4.1 *Let X be a finite set, and let R_E be an extended relation on X satisfying REF and MON. There exists a partial order $\succeq_E$ on $P^0(X)$ satisfying for all x, A, B ,*

$$\begin{aligned} A \succeq_E A \cup x &\Leftrightarrow AR_E x, \text{ and} \\ A \subseteq B &\Rightarrow B \succeq_E A, \end{aligned}$$

if and only if R_E satisfies TRA

Note that in Lemma 4.1, the first condition states that $\succeq_E$ is a (proper) extension of R_E . The second condition is weak monotonicity of $\succeq_E$ with respect to set inclusion.

Proof of Lemma 4.1 Necessity of TRA can be verified as follows. Suppose that $(A \cup y)R_E x$ and $B R_E y$, hence by MON, $(A \cup B \cup y)R_E x$ and $(A \cup B)R_E y$, respectively. By the fact, that $\succeq_E$ is an extension, $A \cup B \succeq_E A \cup B \cup y$ and $A \cup B \cup y \succeq_E A \cup B \cup y \cup x$, hence by transitivity, $A \cup B \succeq_E A \cup B \cup y \cup x$. Finally, by monotonicity with respect to set inclusion and transitivity, $A \cup B \succeq_E A \cup B \cup x$, i.e. $(A \cup B)R_E x$.

In order to verify sufficiency of TRA, define a binary relation $\succeq_E^*$ on $P^0(X)$ as follows. For all A, B ,

$$A \succeq_E^* B \Leftrightarrow \forall y \in B : AR_E y. \quad (4.6)$$

⁵Note that this may serve as an additional justification for property TRA as the natural extension of transitivity for extended relations.

Obviously, since R_E satisfies REF, $\succeq_E^*$ is an extension of R_E that is monotonic with respect to set inclusion. Hence, it suffices to show that $\succeq_E^*$ is transitive. Suppose that for $A, B, C \in P^0(X)$, $A \succeq_E^* B$ and $B \succeq_E^* C$, i.e. $AR_E y$ for all $y \in B$ and $BR_E z$ for all $z \in C$. We have to show that for all $z \in C$, $AR_E z$. This follows at once from MON if $B \subseteq A$. Hence, suppose that $B \setminus A = \{b_1, \dots, b_n\}$ is non-empty. Define $B_0 := A$, and for $i = 1, \dots, n$, $B_i := A \cup \{b_1, \dots, b_i\}$, so that $B_n = A \cup B$. By MON, $BR_E z$ implies $B_n R_E z$ for all $z \in C$. This can be written as

$$(B_{n-1} \cup b_n) R_E z$$

for all $z \in C$. Also, by assumption, $AR_E b_n$. Hence, by TRA, for all $z \in C$, $B_{n-1} R_E z$. Again, this can be written as

$$(B_{n-2} \cup b_{n-1}) R_E z,$$

which together with $AR_E b_{n-1}$ implies $B_{n-2} R_E z$ for all $z \in C$ by TRA. Thus, by induction, $B_0 R_E z$, i.e. $AR_E z$ for all $z \in C$. $\square$

Note that the particular extension $\succeq_E^*$ given by (4.6) is defined in terms of a *dominance* condition: A is weakly preferred to B if and only if every alternative of B is inessential at A . It can be shown that this, in effect, determines the *smallest* extension of R_E to $P^0(X)$, i.e. $\succeq_E^*$ is the intersection of all extensions to $P^0(X)$.

It is easily verified that, given REF, the extension $\succeq_E^*$ defined above satisfies the following property.

Indirect Utility Dominance (IUD) For all A, B ,

$$A \succeq B \Leftrightarrow A \succeq A \cup B$$

Theorem 4.2 (Kreps) *Let X be finite, and let $\succeq$ be a binary relation on $P^0(X)$. There exists a set $\mathcal{R}$ of weak orders on X such that for all A, B ,*

$$A \succeq B \Leftrightarrow \forall R \in \mathcal{R} : A \succeq^R B,$$

if and only if $\succeq$ is a partial order satisfying IUD.

This is the version of Kreps' theorem stated and proved in Puppe [1996, Corollary 4]. For a sketch of the proof of Theorem 4.2 on infinite domains, see Sect. 6. Given this result, the proof of Theorem 4.1 proceeds in the following straightforward manner.

Proof of Theorem 4.1 Necessity of REF, MON and TRA for multi-preference rationalizability is easily verified. For the sufficiency part, let R_E satisfy REF, MON and TRA. By Lemma 4.1 there exists an extension of R_E to $P^0(X)$. The particular extension $\succeq_E^*$ constructed above in addition satisfies IUD. Hence, by Theorem 4.2 there exists a set $\mathcal{R}$ of weak orders on X such that $A \succeq_E^* B$ if and

only if for all $R \in \mathcal{R}$ the corresponding indirect utility preference $\succeq^R$ satisfies $A \succeq^R B$. In particular, $AR_E x$ if and only if, for all $R \in \mathcal{R}$, $AR x$. $\square$

As a corollary of Theorem 4.1 one obtains the following characterization of WEPOs analogous to the characterization of SEPOs provided by Corollary 3.1. Denote by $\mathcal{W}$ the set of all weak orders on X .

Corollary 4.1 *Let X be finite. An extended relation R on X is a WEPO if and only if it is the intersection of all its weak order extensions, i.e.,*

$$R = \bigcap \{Q \in \mathcal{W} : Q \supseteq R\}$$

Note that in our context, binariness has very little to recommend itself. For instance, BIN would yield the following highly restrictive implication: If $\{x, y\} \vdash x$ for all $x, y \in X$, then $A \supset B \Rightarrow A \succ_E^* B$ for all A, B (" $\subset$ " denoting proper inclusion). In terms of the intersection representation, BIN is equivalent to the following condition. Let $\mathcal{R} = \{R_1, \dots, R_n\}$ be a representing family of weak orders as in (4.5). For all A, B and all x ,

$$[\exists R_i \forall a \in A : x P_i a, \text{ and } \exists R_j \forall b \in B : x P_j b] \Rightarrow \exists R_k \forall c \in A \cup B : x P_k c,$$

which seems remarkably unattractive.

5 Properness

The rationalization result for choice functions in terms of SEPOs on the one hand, and the representation of "preference for opportunities" in terms of WEPOs on the other, raise the question of the precise relationship between reflexive and irreflexive EPOs. This issue is addressed in this section. In particular, an appropriate additional condition of "strict properness" is shown to provide a canonical link between SEPOs and WEPOs. Furthermore, it is argued that the notion of properness may serve as a foundation for the concept of "choice-based" preference.

5.1 Strict Properness

Given the concept of an extended relation as introduced in this paper, the following definition seems to be natural.

Definition 5.1 For any extended relation Q on X , define its *irreflexive component* P_Q as follows. For all x, A ,

$$AP_Q x :\Leftrightarrow (A \setminus x)Qx$$

Also, define the *reflexive closure* R_Q of Q by

$$AR_Q x :\Leftrightarrow [AQx \text{ or } x \in A],$$

for all x, A .

It is easily verified that, given monotonicity of Q , P_Q is the greatest irreflexive subrelation of Q . Similarly, R_Q is the smallest reflexive extension of Q . In particular, $P_Q = Q$ if Q satisfies IRR, and $R_Q = Q$ if Q satisfies REF. The operations of taking the irreflexive component and the reflexive closure are mutually inverse in the following sense.

Lemma 5.1 If R is a reflexive and monotone extended relation on X , then $R = R_{(P_R)}$. Similarly, if P is an *irreflexive* and monotone extended relation on X , then $P = P_{(R_P)}$.

The proof is straightforward and therefore omitted. It is also easily verified that for any SEPO P , the reflexive closure R_P is a WEPO. An analogous statement for WEPOs is, however, not true as the following example demonstrates.

Example 5.1 Let $X = \{x, y, z\}$, and a WEPO R on X as follows. For all x, A ,

$$ARx \Leftrightarrow [A = \{x\} \text{ or } \#A \geq 2]$$

It can be verified that the irreflexive component P_R of R is given by

$$AP_Rx \Leftrightarrow [A = \{x, y, z\} \text{ or } (\#A = 2 \text{ and } x \notin A)]$$

However, P_R is not transitive as can be verified by choosing $A = B = \{x, z\}$ in the definition of TRA.

Example 5.1 thus shows that additional conditions are needed in order to guarantee transitivity of the irreflexive component of a WEPO. It turns out that the appropriate condition is as follows.

Strict Properness (SPR) For all $A \cup x, A \cup y, x \neq y$,

$$[(A \cup x)Qy \text{ and } (A \cup y)Qx] \Rightarrow [A \neq \emptyset \text{ and } AQx].$$

The following lemma clarifies the interrelation of SPR and the basic properties of extended partial orders. In particular, it shows that any SEPO automatically satisfies SPR.

Lemma 5.2 For any extended relation, (i) IRR and TRA jointly imply SPR, and (ii) MON and SPR jointly imply TRA.

Proof (i) Let $A \cup x, A \cup y$ with $x \neq y$ be given such that $(A \cup x)Qy$ and $(A \cup y)Qx$. First, it is easily established that by IRR and TRA, $A \neq \emptyset$. By IRR, $[(A \cup x) \setminus y]Qy$ and $[(A \setminus x) \cup y]Qx$, hence by TRA, $[(A \setminus x) \cup ((A \cup x) \setminus y)]Qx$. This implies $(A \cup x)Qx$, hence AQx by IRR.

(ii) Let $A \cup y, B$ and x be given such that $(A \cup y)Qx$ and BQy . If $x = y$, $(A \cup B)Qx$ by MON. Hence, suppose $x \neq y$. By MON, $(A \cup B \cup y)Qx$ and $(A \cup B \cup x)Qy$, hence by SPR, $(A \cup B)Qx$.

□

Theorem 5.1 *If $\mathbf{P}$ is a SEPO, then R_P is a WEPO satisfying SPR. Conversely, if R is a WEPO satisfying SPR, then P_R is a SEPO.*

Proof The first part of Theorem 5.1 is easily established using Lemma 5.2(i) and the fact that R_Q satisfies SPR whenever Q does. Consider the second part. Irreflexivity and monotonicity of P_R are obvious. In order to verify transitivity, observe that P_R satisfies SPR, since R does. Hence, the claim follows from Lemma 5.2(ii). $\square$

5.2 Strict Properness and Choice

In this subsection, it is shown that the concept of strict properness allows for a canonical definition of "revealed weak preference." We will also argue that properness is the "right" condition in order to demarcate the notion of "choice-based" preference. As has been argued in Nehring [1996], condition ρ provides a canonical link between choice and revealed strict preference in terms of SEPOs. Consequently, one may use the correspondence between SEPOs and strictly proper WEPOs established in Theorem 5.1 in order to define revealed weak preference. Given a choice function $C : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$, define the revealed weak extended preference relation R_C as follows. For all x and all $A \in \mathcal{F}(X)$,

$$(p') \quad AR_C x \Leftrightarrow C(A \cup x) \subseteq A.$$

Intuitively, condition ρ' rationalize? weak superiority of sets over elements, whereas condition ρ rationalizes inferiority of non-chosen alternatives. It is easily verified that if R and C satisfy ρ' then P_R and C satisfy p . Conversely, if P and C satisfy p then R_P and C satisfy p' . The following result is thus an immediate corollary of Theorems 3.1 and 5.1.

Corollary 5.1 *Let R_C be defined from a choice function C according to ρ' . Then R_C is a WEPO satisfying SPR if and only if C satisfies α and η .*

This result suggests an identification of *choice basedness* with strict properness, at least if one is willing to accept α and η as basic conditions of consistent choice. In order to shed further light on the relation between choice and (weak) preference, consider again condition ρ' which describes how revealed weak preference can be canonically obtained from given choice behaviour. One may ask, conversely, what choice function is generated by a given WEPO R . Consider the following definition. Given a WEPO R let for any A ,

$$C_R(A) := \bigcap \{B \subseteq A : BRx \text{ for all } x \in A\} \quad (5.1)$$

Hence, $C_R(A)$ is the intersection of all "dominating" subsets of A . Thus, one would like to interpret the choice function C_R as describing, for each A , the minimal set that a decision maker is "unambiguously entitled" to confine himself

to in choosing given his preferences. As the following result shows, such an interpretation is warranted if and only if R is strictly proper.

Theorem 5.2 *Let X be finite, and let R be a monotone and reflexive extended relation on X . The choice function C_R satisfies $C_R(A) \neq \emptyset$ for all A , and $C_R(A)Rx$ for all $x \in A$, if and only if R satisfies SPR.*

Proof Necessity of SPR can be verified as follows. Let $x \neq y$ be given such that $(A \cup x)Ry$ and $(A \cup y)Rx$. By definition of C_R ,

$$C_R(A \cup \{x, y\}) \subseteq (A \cup x) \cap (A \cup y) = A.$$

In particular, $A \neq \emptyset$ and, as a consequence of $C_R(A \cup \{x, y\})Rz$ for all $z \in A \cup \{x, y\}$, ARz for all $z \in A \cup \{x, y\}$ by MON. In particular, ARx as required by SPR.

Conversely, let R satisfy SPR. In order to show the desired conclusion it suffices, by induction, to verify the following. For all non-empty $B, C \subseteq A$ such that BRz and CRz for all $z \in A$, the intersection $B \cap C$ is non-empty and satisfies $(B \cap C)Rz$ for all $z \in A$. This is trivial if one of the sets, B or C , is equal to A . Hence, we may assume $B \neq A$. In this case, the set $\{x_1, \dots, x_m\} =: A \setminus (B \cap C)$ is non-empty. We will show that for any subset $D \subseteq \{x_1, \dots, x_m\}$, $(A \setminus D)Rz$ for all $z \in A$. This will be done by induction on the number of elements of D . First, consider a one-element subset $D^{(1)} = \{x_i\}$. By REF, one has $[(A \setminus x_i) \cup x_i]Rz$ for any $z \in A \setminus x_i$. Also, by MON, for any $z \in A \setminus x_i$, $[(A \setminus x_i) \cup z]Rx_i$ since either $B \subseteq A \setminus x_i$, or $C \subseteq A \setminus x_i$. Hence, by SPR, $A \setminus x_i$ is non-empty and $(A \setminus x_i)Rx_i$. Now assume the claim is verified for all subsets $D^{(k)}$ containing exactly k elements and consider any set $D^{(k+1)} \subseteq \{x_1, \dots, x_m\}$ that contains $k+1$ elements. Let x_i, x_j be any two different elements of $D^{(k+1)}$. By induction hypothesis, $[(A \setminus D^{(k+1)}) \cup x_i]Rx_j$ and $[(A \setminus D^{(k+1)}) \cup x_j]Rx_i$. Hence, by SPR, $A \setminus D^{(k+1)} \neq \emptyset$ and $(A \setminus D^{(k+1)})Rx_i$. Since x_i was an arbitrary element of $D^{(k+1)}$, this shows $(A \setminus D^{(k+1)})Rz$ for all $z \in A$ by REF. Consequently, $B \cap C \neq \emptyset$ and $(B \cap C)Rz$ for all $z \in A$, since $B \cap C = A \setminus \{x_1, \dots, x_m\} = A \setminus D^{(m)}$. $\square$

The fact that SPR is a necessary condition for C_R to be well-defined can be further illustrated by considering the WEPO of Example 5.1 above.

Example 5.1 (continued) Let R be the WEPO on $X = \{x, y, z\}$ as defined in Example 5.1 above. It is easily verified that according to Theorem 4.1, R can be represented as follows. For all w, A ,

$$ARw \Leftrightarrow \forall i = 1, 2, 3 : \max_{v \in A} u_i(v) \geq u_i(w),$$

with the following utility functions $u_{1,2,3}$ on $\{x, y, z\}$:

	u_1	u_2	u_3
x	0	1	1
y	1	0	1
z	1	1	0

Clearly, the WEPO R as defined is not strictly proper. What would be the "reasonable" chosen sets induced by the WEPO R ? Obviously, any two-element subset of $\{x, y, z\}$ dominates each of its single elements, while no singleton dominates any other element. Hence, in accordance with definition (5.1), $C_R(B) = B$ for each two-element subset B of $\{x, y, z\}$. On the other hand, choice from $\{x, y, z\}$ is not as straightforward. While for any two-element subset B , BRw for all w , there is no (unambiguously) smallest dominating set, i.e. $C_R(\{x, y, z\}) = \emptyset$. One might suggest to define the chosen set from $\{x, y, z\}$ as $\{x, y, z\}$ itself. However, that would inevitably result in the inclusion of "redundant" alternatives. The example thus clearly demonstrates that indeed not all WEPOs are canonically related to consistent choice behaviour.

The following result lends further support to the proposed conceptual identification of "choice-basedness" and strict properness, showing that under SPR condition ρ' and definition (5.1) describe *mutually inverse* operations.

Fact 5.1 Let R be a WEPO satisfying SPR. Then, $R_{(C_R)} = R$, and $C_{(R_C)} = C$.

Proof To verify the first part, suppose that $AR_{(C_R)}x$, i.e. by condition ρ' , $C_R(A \cup x) \subseteq A$. By Theorem 5.2, $C_R(A \cup x)Rz$ for all $z \in A \cup x$, in particular, $C_R(A \cup x)Rx$. Hence, by MON, ARx . Conversely, suppose that ARx . Then, by REF, ARz for all $z \in A \cup x$. This implies $C_R(A \cup x) \subseteq A$, and hence $AR_{(C_R)}x$ by condition ρ' .

For the second part, we have to verify that for all A , $C(A) = C_{(R_C)}(A)$. This is trivial if A contains one single element. Hence, assume that A has at least two elements. First, let $x \in C(A)$. This implies $C((A \setminus x) \cup x) \not\subseteq A \setminus x$, hence by ρ' , not $[(A \setminus x)R_Cx]$. By Theorem 5.2, $C_{(R_C)}(A)R_Cx$, hence by MON, $C_{(R_C)}(A) \not\subseteq A \setminus x$, i.e. $x \in C_{(R_C)}(A)$. Conversely, let $x \notin C(A)$. This implies $C((A \setminus x) \cup x) \subseteq A \setminus x$, hence by ρ' , $(A \setminus x)R_Cx$. By definition (5.1), one thus obtains $C_{(R_C)}(A) \subseteq A \setminus x$, i.e. $x \notin C_{(R_C)}(A)$. $\square$

5.3 Weak Properness

The condition of strict properness defined above entails a condition of *antisymmetry*. To see this, consider first the case of binary extended relations. First, observe that P_Q and R_Q are binary whenever Q is binary. Let R be a binary WEPO with base relation $R_b \subseteq X \times X$. Then, the base relation $(P_R)_b$ of P_R is given by $(P_R)_b = R_b \setminus \{(x, x) : x \in X\}$. Clearly, in general the relation $(P_R)_b$ is

irreflexive, but note that it is not necessarily antisymmetric. Indeed, it is easily verified that for a binary WEPO R , the base relation is antisymmetric if and only if R satisfies SPR. Hence, the notion of a strictly proper WEPO may be thought of as generalizing the notion of an antisymmetric partial order, i.e. a (reflexive) strict partial order. **Furthermore**, this suggests to think of SPR as being composed of the following two conditions.

Weak Properness (WPR.) For all $A \cup x, A \cup y$ such that not $[xQy \text{ and } yQx]$,

$$[(A \cup x)Qy \text{ and } (A \cup y)Qx] \Rightarrow [A \neq \emptyset \text{ and } AQx].$$

Antisymmetry (ANT) For all x, y ,

$$[xQy \text{ and } yQx] \Rightarrow x = y.$$

For the record, we note:

Fact 5.2 *SPR is equivalent to the conjunction of WPR and ANT*

As will be shown in the next subsection, WPR is a condition that deserves interest on its own right (although it does not seem sufficient for a canonical link between SEPOs and WEPOs). Observe that in contrast to SPR, any binary transitive extended relation automatically satisfies WPR. Hence, unlike SPR its weakening WPR is a condition that has bite only in the general framework of *extended* relations.

5.4 Multi-Preference Rationalizability of Proper WEPOs

Given the multi-preference representation of a WEPO by set of weak orders R on X , a natural question. is how the condition of properness is reflected in properties of the representing family of weak orders. Based on the analysis in Nehring and Puppe [1996], we provide a complete answer in the case of finite X . In order to formulate the result, consider first the following definition.

Definition 5.1 (i) Say that, an extended relation R admits a *linear* representation if and only if there exists a set $\mathcal{R}$ of linear orders (i.e. antisymmetric weak orders) such that R satisfies (4.5) with respect to $\mathcal{R}$.

(ii) Similarly, R admits an *effectively linear* representation if and only if there exists a representing family $\mathcal{R} = \{R_1, \dots, R_n\}$ of weak orders with the property that any indifference occurring in one ordering $R_i \in \mathcal{R}$ is shared by all orderings in $\mathcal{R}$, i.e. $xI_i y$ for some i implies $xI_j y$ for all j .

(iii) Finally, R admits a *convex* representation if and only if there exists a representing family $\mathcal{R} = \{R_1, \dots, R_n\}$ of weak orders and, for each i , a utility function u_i representing R_i such that any element of the convex hull of $\mathcal{U} = \{u_1, \dots, u_n\}$ also represents one of the orderings in $\mathcal{R}$.

Theorem 5.3 *Let X be finite, and let R be a WEPO on X . Then,*

- (i) *R has a linear representation if and only if R satisfies SPR*
- (ii) *R has an effectively linear representation if and only if R satisfies WPR.*
- (iii) *R has a convex representation if and only if R satisfies WPR.*

The entailed equivalence of effective linearity and convexity of a representation has been established in Nehring and Puppe [1996] in a slightly different framework. Specifically, in that paper it has been shown that in the context of ranking opportunity sets, effective linearity and convexity of a multi-preference representation are each equivalent to a condition of "Irrelevance of Inessential Elements" (IIE). It can easily be checked that when translated into a condition on the corresponding WEPO, IIE corresponds to our condition WPR. Part (i) of Theorem 5.3 is, in effect, a special case of part (ii), noting that effective linearity of a representation entails its linearity if R is antisymmetric. Alternatively, Theorem 5.3(i) can be deduced from Theorems 3.1 and 5.1 above. Indeed, if R is a WEPO satisfying SPR, then by Theorem 5.1, its irreflexive component P_R is a SEPO. Hence, by Theorem 3.1 (or, Corollary 3.1) there exists a representation of P_R by a set of (irreflexive) linear orderings. It is then easily verified that the reflexive closures of these linear orderings represent R in the sense of condition (4.5).

Multi-preference representations of extended orders allow to rationalize preference incompleteness in terms of "unresolvedness" of preference due to suspension of judgement between a set of "relevant" complete preference orderings that induces a corresponding suspension of judgement between the alternatives themselves. Conversely, asserted weak preference is rationalized as *definite* (though possibly "disjunctive") preference according to

$$ARx \Leftrightarrow \forall R_i \in \mathcal{R} \exists y_i \in A : y_i R_i x. \quad (5.2)$$

Note that the non-binarity of R can be accounted for in terms of (5.2) as a failure of the " $\forall$ " and " $\exists$ " quantifiers to interchange. Conceptually, the " $\Leftarrow$ " implication of (5.2) seems not entirely unproblematic. In particular, one might argue that even if x is dominated by some y ; for any particular R_i , x may nonetheless be a potentially superior *compromise* choice. In Example 5.1, for instance, z is clearly the best choice with respect to the subset $\{u_1, u_2\}$ of utility functions. This skepticism can be addressed by requiring the set of rationalizing orderings $\mathcal{R}$ to be "closed under compromise," i.e. to be convex (see Nehring [1996] for an extensive argument). In Example 5.1, z is uniquely best with respect to any compromise utility function $v = \sum_i \alpha_i u_i$ with $\sum_i \alpha_i = 1$ and $0 \leq \alpha_3 < \min\{\alpha_1, \alpha_2\}$, and can thus not be viewed as genuinely dominated by the set $\{x, y\}$. While it has been shown in Nehring [1996] that convexity is without loss of generality for SEPOs, Theorem 5.3(iii) shows convexity to entail significant restrictions for WEPOs. It seems rather remarkable that, up to a

different stance toward indifferences, closedness under compromise leads to the same restriction as the requirement of choice-basedness in the sense of Theorem 5.2.

6 Non-finite Domains

In this section, we address the problem of how the main results of this paper generalize to the case of arbitrary, possibly non-finite, domains. Specifically, we will consider the results on multi-preference rationalizability of extended relations, i.e. Theorems 4.1, 5.3(i) and 5.3(ii).⁶ First, consider the multi-preference representation for WEPOs established in Theorem 4.1. In order to prove a corresponding result for arbitrary domains, we first need to state a version of Kreps' theorem applicable also to infinite domains. As the following example demonstrates, the multiple preference representation according to Theorem 4.2 requires additional restrictions in that case.

Example 6.1 Let $X = \mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers. Define a binary relation $\succeq$ on $P^0(X)$ as follows. For all A, B ,

$$A \succeq B :\Leftrightarrow [A \text{ is infinite, or } B \text{ is finite}].$$

Obviously, $\succeq$ is reflexive and transitive, i.e. a partial order. Also, it is easily verified that $\succeq$ satisfies IUD. However, there does not exist a family $\mathcal{R}$ of weak orders on X such that $\succeq$ is the intersection of all corresponding indirect utility preferences as required by Theorem 4.2. To see this, assume to the contrary that there would exist such a family $\mathcal{R} = \{R_i : i \in \mathcal{I}\}$ for some index set $\mathcal{I}$. By definition, $1 \succeq w$ for all $w \in \mathbb{N}$. Hence, for all $i \in \mathcal{I}$, and all $w \in \mathbb{N}$, $1 R_i w$. This implies $1 \succ^{R_i} \mathbb{N}$ for all i by definition of $\succeq^{R_i}$. Consequently, one would obtain $1 \succeq \mathbb{N}$ which is false by assumption.

As it turns out, the existence of a multi-preference representation can be guaranteed by the following “continuity” condition. Let $\mathcal{C} \subseteq P^0(X)$ be a *chain* of subsets of X , i.e. a family of subsets that are completely ordered by set inclusion. For all A and all chains $\mathcal{C}$,

$$\forall C \in \mathcal{C} : A \succeq C \Rightarrow A \succeq \bigcup \mathcal{C}. \quad (6.1)$$

Note that on a finite domain any binary relation $\succeq$ (vacuously) satisfies (6.1)

⁶We do not consider the problem of existence of a convex representation as in Theorem 5.3(iii), since in this context additional difficulties arise. Indeed, in order to define the concept of convexity on arbitrary domains one would need additional conditions that guarantee the existence of a utility representation for a given weak order. Therefore, such an investigation is beyond the scope of the present paper.

Theorem 4.2' (Kreps, infinite version) Let $\succeq$ be a binary relation on $P^0(X)$. There exists a set $\mathcal{R}$ of (possibly infinitely many) weak orders on X such that for all A, B ,

$$A \succeq B \Leftrightarrow \forall R \in \mathcal{R} : A \succeq^R B,$$

if and only if $\succeq$ is a partial order that is continuous in the sense of (6.1) and satisfies IUD.

Sketch of proof of Theorem 4.2' Necessity of the continuity condition (6.1) is easily verified along the lines of Example 6.1. For the sufficiency part, consider the following mapping $f : P^0(X) \rightarrow P^0(X)$. For all A ,

$$f(A) := \bigcup_{B: \succeq B} B$$

Using condition IUD and the continuity condition (6.1), one easily verifies that, for all A , $A \succeq f(A)$. This implies that, for all A, B , $A \succeq B \Leftrightarrow f(B) \subseteq f(A)$. Using this, one can further verify that, for all A , $f(f(A)) = f(A)$, hence the sets $f(A)$, $A \in P^0(X)$, are precisely the *fixed points* of the mapping f . Let Γ_{max} denote the set of all maximal chains in the set $\{f(A) : A \in P^0(X)\}$ of fixed points of f , where a maximal chain is a chain that is not proper subset of any other chain. For each $\mathcal{H} \in \Gamma_{max}$ define a weak order on $R_{\mathcal{H}}$ on X as follows. For all $x, y \in X$,

$$x R_{\mathcal{H}} y \Leftrightarrow [\forall H \in \mathcal{H} : x \in H \Rightarrow y \in H]. \quad (6.2)$$

It can be verified that for all A, B ,

$$A \succeq B \Leftrightarrow \forall R_{\mathcal{H}} \in \Gamma_{max} : A \succeq^{R_{\mathcal{H}}} B,$$

i.e. the family $\{R_{\mathcal{H}} : \mathcal{H} \in \Gamma_{max}\}$ provides the desired representation. $\square$

Remark Note that the representing set $\mathcal{R}$ in Theorems 4.2 or 4.2' is not uniquely determined by the ranking $\succeq$. The specific family of weak orders constructed in the proof of Theorem 4.2' will be referred to as the *maximal chain representation* of $\succeq$ (cf. Kreps [1979]).

Consider now the problem of multi-preference rationalizability of a given WEPO R on X . In order to apply Kreps' theorem, one has to extend R to a continuous partial order that satisfies IUD. As the following example suggests, the existence of such an extension requires additional restrictions on infinite domains.

Example 6.2 Let $X = N \cup \{x_0\}$, where N is the set of natural numbers, and x_0 is some object not contained in N , e.g. one may choose $x_0 = \infty$. Define an extended relation R on X as follows. For all $i \in N$, and all $A \in P^0(X)$, $A R_i$. Furthermore, for all $A \in P^0(X)$,

$$A R x_0 \Leftrightarrow [x_0 \in A \text{ or } A \text{ is infinite}].$$

It is easily verified that R satisfies REF, MON and TRA, i.e. R is a WEPO. However, the canonical extension $\succeq^*$ of R according to (4.6) is not transitive. For instance, $\{1\} \sim^* N$ and $N \sim^* N \cup \{x_0\}$, but $N \cup \{x_0\} \succ^* \{1\}$.

In the infinite case, transitivity of the canonical extension (4.6) is guaranteed by the following "regularity" condition.

Regularity (REG) Let C be a chain. Suppose that for all $C \in \mathcal{C}$ and all $y \in \bigcup \mathcal{C}$, CRy . Then, for all $z \in X$,

$$\bigcup CRz \Rightarrow \exists C \in \mathcal{C} : CRz$$

Note that as the continuity condition (6.1), REG is vacuously satisfied in the finite case.

Lemma 6.1 Let R be an extended relation on X satisfying REF and MON. There exists a partial order $\succeq$ on $P^0(X)$, continuous and monotone with respect to set inclusion, that extends R if and only if R satisfies TRA and REG.

Proof Necessity of TRA follows as in the finite case. Necessity of REG can be verified as follows. Let C be a chain. Choose a well-ordering of $\bigcup \mathcal{C}$ so that $\bigcup \mathcal{C} = \{y_i : i < \nu\}$ for some (possibly infinite) ordinal ν . Take any $C \in \mathcal{C}$. By assumption, CRy_i for all $i < \nu$. First, we show by transfinite induction that this implies $C \succeq C \cup \bigcup \mathcal{C}$, and hence $C \succeq \bigcup \mathcal{C}$. Clearly, $C \succeq C \cup y_1$ since $\succeq$ is an extension of R . Suppose it has been shown that $C \succeq C \cup \{y_i : i < \mu + 1\}$ for some $\mu < \nu$. By MON, $CRy_{\mu+1}$ implies $(C \cup \{y_i : i < \mu + 1\})Ry_{\mu+1}$, i.e. $C \cup \{y_i : i < \mu + 1\} \succeq C \cup \{y_i : i \leq \mu + 1\}$. Hence, by transitivity $C \succeq C \cup \{y_i : i \leq \mu + 1\}$. Next, suppose it has been shown that $C \succeq C \cup \{y_i : i < \lambda\}$ for some limit ordinal $\lambda < \nu$. Then, by continuity, $C \succeq C \cup \bigcup \{y_i : i < \lambda\}$. Together with $C \succeq C \cup y_\lambda$ this implies $C \succeq C \cup \{y_i : i \leq \lambda\}$. Thus, by transfinite induction, $C \succeq \bigcup \mathcal{C}$. Now suppose, according to REG, that $\bigcup CRz$, i.e. $\bigcup \mathcal{C} \succeq \bigcup \mathcal{C} \cup z$. For any $C \in \mathcal{C}$, $C \succeq \bigcup \mathcal{C}$ by the first part, hence by transitivity, $C \succeq \bigcup \mathcal{C} \cup z$. By monotonicity of $\succeq$, $C \succeq C \cup z$, i.e. CRz .

In order to verify sufficiency of TRA and REG, consider the canonical extension $\succeq^*$ of R defined, as in (4.6), by

$$A \succeq^* B :\Leftrightarrow \forall y \in B : ARy,$$

for all A, B . By definition, $\succeq^*$ is an extension that is monotone with respect to set inclusion and continuous in the sense of (6.1). Hence, it suffices to verify transitivity. Let $A \succeq^* B$ and $B \succeq^* C$, i.e. ARy for all $y \in B$ and BRz for all $z \in C$. We have to show ARz for all $z \in C$. As in the proof of Lemma 4.1 this follows at once from MON if $B \subseteq A$. Hence, suppose that $B \setminus A$ is non-empty. Choose a well-ordering of $B \setminus A$ so that $B \setminus A = \{b_i : i < \nu\}$ for some ordinal ν . Define $B_0 := A$ and, for each $\mu \leq \nu$, $B_\mu := A \cup \{b_i : i < \mu\}$ so that $B_\nu = A \cup B$. The proof proceeds by transfinite (downward) induction. Let $z \in C$. Clearly,

$B_\nu Rz$ by MON. Suppose that for some ordinal μ , $B_{\mu+1}Rz$. Then one concludes as in the proof of Lemma 4.1 that TRA implies $B_\mu Rz$. Next, suppose that $B_\lambda Rz$ for some limit ordinal λ . Since for any $\mu \leq \nu$, $A \subseteq A_\mu$, one has $B_\mu Ry$ for any $y \in A \cup B$ by MON. Hence, by condition REG, $B_\mu Rz$ for some $\mu < \lambda$. Therefore, by induction $B_0 Rz$, i.e. ARz . $\square$

Clearly, as in the finite case property REF implies that the canonical extension $\succeq^*$ satisfies IUD. Combining Theorem 4.2' and Lemma 6.1 one thus obtains the following characterization of multi-preference rationalizability of extended relations.

Theorem 6.1 *Let R be an extended relation on an arbitrary domain X . R is multi-preference rationalizable (in the sense of (4.5)) if and only if R satisfies REF, MON, TRA and REG, i.e. if and only if R is a regular WEPO.*

The following corollary is analogous to Corollary 4.1

Corollary 6.1 *An extended relation R on X is a regular WEPO if and only if it is the intersection of all its weak order extensions.*

Consider now the statements (i) and (ii) of Theorem 5.3. Let R be a WEPO, and denote by $\succeq^*$ the canonical extension of R as defined in the proof of Lemma 6.1, i.e. $A \succeq^* B \Leftrightarrow \forall y \in B : ARy$. In the finite case, the proof of Theorem 5.3(i) and (ii) consists in showing that R satisfies SPR (WPR) if and only if the maximal chain representation of $\succeq^*$ is linear (effectively linear). Hence, in the finite case, the existence of a linear (effectively linear) representation is equivalent to linearity (effective linearity) of the maximal chain representation. However, this is no longer true for arbitrary domains as the following example shows.

Example 6.3 Let $X = \mathbb{N} \cup \{x_1, x_2\}$, where $x_1, x_2 \notin \mathbb{N}$. For each $i \in \mathbb{N}$ define a weak order R_i by the following utility function $u_i : X \rightarrow \mathbb{R}$,

$$u_i(z) = \begin{cases} 3 & \text{if } z = i \\ 2 & \text{if } z = x_2 \\ 1 & \text{if } z = x_1 \\ \frac{1}{j} & \text{if } z = j \text{ with } j \in \mathbb{N}, j \neq i \end{cases}$$

Define a regular WEPO by $R := \bigcap_i R_i$. Clearly, since all orderings R_i are linear, R satisfies SPR. However, the maximal chain representation of the canonical extension $\succeq^*$ is not linear, nor even effectively linear. To see this, observe that the set of all fixed points of the corresponding mapping f as defined in the proof of Theorem 4.2' is $\{\mathbb{N} \cup \{x_1, x_2\}\} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, where

$$\begin{aligned} \mathcal{F}_1 &:= \{A : A \subset \mathbb{N}, A \neq \emptyset\}, \\ \mathcal{F}_2 &:= \{A \cup \{x_1\} : A \subset \mathbb{N}\}, \\ \mathcal{F}_3 &:= \{A \cup \{x_1, x_2\} : A \subset \mathbb{N}\} \end{aligned}$$

(Note that “ $\subset$ ” denotes *proper* inclusion.) Using this, it is easily verified that the maximal chain representation contains orderings for which x_1 and x_2 are indifferent, as well as orderings for which x_2 is strictly better than x_1 . Consequently, the maximal chain representation is not (effectively) linear.

Linearity and effective linearity of the maximal chain representation can be characterized using the following stronger notion of regularity.

Strong Regularity (SREG) For all chains C and all z ,

$$\bigcup C R z \Rightarrow \exists C \in \mathcal{C} : C R z$$

In the following theorem, by the maximal chain representation of a WEPO R we mean the maximal chain representation of the corresponding canonical extension $\succeq^*$ of R to $P^0(X)$.

Theorem 6.2 *Let R be a WEPO on an arbitrary domain X*

- (i) *The maximal chain representation of R is linear if and only if R satisfies SREG and SPR.*
- (ii) *The maximal chain representation of R is effectively linear if and only if R satisfies SREG and WPR.*

Proof We only prove part (ii). From this, part (i) follows at once, noting that for an antisymmetric WEPO, effective linearity and linearity of a representation coincide.

Necessity of WPR is easily verified. Necessity of SREG can be verified as follows. Let C be any chain of non-empty subsets of X . Suppose by way of contraposition, that for all $C \in \mathcal{C}$, $C \cup z \succ^* C$, where $\succeq^*$ is the canonical extension of R (cf. (4.6)). Denote by f the corresponding mapping as defined in the proof of Theorem 4.2'. Clearly, for all $C \in \mathcal{C}$,

$$f(C) \cup z \succ^* f(C). \quad (6.3)$$

The set $\{f(C) : C \in \mathcal{C}\}$ forms a chain in X , and hence can be extended to a maximal chain, denoted by $\mathcal{H}$. Let $R_{\mathcal{H}}$ be the weak order corresponding to $\mathcal{H}$ according to (6.2). By (6.3), $z \notin f(C)$ for all $C \in \mathcal{C}$, hence by definition of $R_{\mathcal{H}}$, $z R_{\mathcal{H}} x$ for all $x \in \bigcup \{f(C) : C \in \mathcal{C}\}$. Suppose that for some $C \in \mathcal{C}$ and some $x \in f(C)$, $x R_{\mathcal{H}} z$. Then, $x I_{\mathcal{H}} z$ and by effective linearity, x and z must be indifferent with respect to every ordering in the maximal chain representation. However, this is not possible by (6.3). Consequently, one must have $z P_{\mathcal{H}} x$ for all $x \in \bigcup \{f(C) : C \in \mathcal{C}\}$. In particular, $z P_{\mathcal{H}} x$ for all $x \in \bigcup \mathcal{C}$, and hence $\bigcup \mathcal{C} \cup z \succ^* \bigcup \mathcal{C}$.

In order to verify sufficiency of SREG and WPR, consider any maximal chain $\mathcal{H}$ of fixed points of the corresponding mapping f . For each $H \in \mathcal{H}$ define a set $H_{<} := \{H' : H' \in \mathcal{H}, H' \subset H\}$. Hence $H_{<}$ is the set of all predecessors of H in

$\mathcal{H}$. Let $H^- := \bigcup H_{<}$. Effective linearity of the maximal chain representation is established by showing that for all $H \in \mathcal{H}$, $\{v, w\} \subseteq H \setminus H^-$ implies $[vRw$ and $wRv]$. First, it is shown that for all $H \in \mathcal{H}$, $H^- \in \mathcal{H}$. Indeed, by SREG, H^- must be a fixed point of f , hence by maximality of the chain, $H^- \in \mathcal{H}$. Note that either $H^- = H$, or H^- is the immediate predecessor of H in $\mathcal{H}$. Suppose that $\{v, w\} \subseteq H \setminus H^-$, and assume by way of contradiction that not $[vRw$ and $wRv]$. Consider $H_1 := f(H^- \cup v)$ and $H_2 := f(H^- \cup w)$. Clearly, $H^- \subset H_1, H_2 \subseteq H$, hence by maximality of the chain, $H_1 = H_2 = H$. This implies $(H^- \cup v)Rw$ and $(H^- \cup w)Rv$, and therefore by WPR, H^-Rv which contradicts the fact that H^- is a fixed point of f . Consequently, vRw and wRv . This immediately implies effective linearity of the maximal chain representation. $\square$

Remark Note that Theorem 6.2 entails Szpilrajn's [1930] well-known extension theorem as a straightforward corollary, since both SREG and WPR are implications of BIN. Also observe that Theorem 6.2 provides only sufficient conditions for the *existence* of a linear (effectively linear) representation of a WEPO on an arbitrary domain. Indeed, as Example 6.3 demonstrates, condition SREG is not necessary for the existence of such a representation. It is conjectured that the key to existence of a linear representation is again the weaker notion of regularity, condition REG.

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